Quantum invariants from unrestricted quantum groups

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UNC Representation Theory Seminar

Acknowledgements

- Thank you to Shrawan Kumar for inviting me to give this talk
- I will be talking about a research program due to a lot of people, with some of my contributions towards the end. There are specific citations throughout the slides.

Reminders

- Please interrupt me! I would rather describe a few things well than many things poorly.
- I will post these slides at esselltwo.com/talks

Plan of the talk

1. Overview/reminder of Reshetikhin-Turaev construction for links
2. RT for unrestricted quantum groups at roots of unity (abelian version)
3. RT for unrestricted quantum groups at roots of unity (nonabelian version) and geometric applications

Quantum groups and quantum invariants

Quantum $\mathfrak{sl}_2$

Definition

Quantum $\mathfrak{sl}_2$ is the algebra $\mathcal{U}_q = \mathcal{U}_q(\mathfrak{sl}_2)$ over $\mathbb{C}[q, q^{-1}]$ with generators $K^{\pm 1}, E, F$ and relations

$$KE = q^2 EK \qquad KF = q^{-2} FK \qquad EF - FE = (q - q^{-1})(K - K^{-1})$$

Idea

This is a non-commutative, non-cocommutative version of the universal enveloping algebra of $\mathfrak{sl}_2$, with $K = q^H$. When $q \to 1$ we recover $\mathfrak{sl}_2$.

An unusual normalization

Warning

Usually we set

$$EF - FE = \frac{K - K^{-1}}{q - q^{-1}}$$

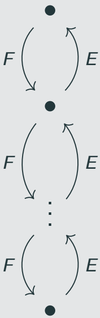
instead. These equivalent algebras whenever $q - q^{-1} \neq 0$.

- The usual normalization gives a non-commutative, cocommutative universal enveloping algebra at $q = 1$
- Ours is so that $\mathcal{U}_1(\mathfrak{sl}_2)$ is a commutative and non-cocommutative Hopf algebra, i.e. the algebra of functions on an algebraic group. Will be important later!
- This is the Kac-de Concini form of the quantum group

Representations of $\mathcal{U}_q$

Fact

For q generic (not a root of unity) any finite-dimensional weight module of dimension N looks like



$$q^\lambda = q^{N-1}$$

$$q^{N-3}$$

$$\vdots$$

$$q^{-(N-1)}$$

- Very similar to $\mathfrak{sl}_2$. Tensor product multiplicities are the same as well.
- What was the point of introducing $\mathcal{U}_q$?
- Because $\mathcal{U}_q$ is not cocommutative, $\tau(x \otimes y) = y \otimes x$ is *not* a $\mathcal{U}_q$-module map

$$V \otimes W \to W \otimes V$$

- Instead there are more interesting ones!

The universal R-matrix

Definition

The universal R-matrix is

$$\mathbf{R} = q^{H \otimes H/2} \sum_{n=0}^{\infty} c_n E^n \otimes F^n \in \mathcal{U}_q \otimes \mathcal{U}_q$$

for some coefficients c_n.

This is an infinite series! Really lives in a certain completion of $\mathcal{U}_q^{\otimes 2}$.
(Another way is to work in power series in $\hbar$, with $q = e^{\hbar}$.)

To fix

On any finite-dimensional $\mathcal{U}_q$-module E and F act nilpotently and H is diagonalizable so the action of $\mathbf{R}$ is well-defined.

The braiding

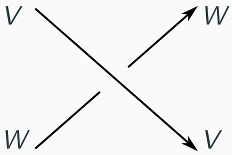
Proposition

For any $\mathcal{U}_q$-modules V, W, the braiding is

$$c_{V,W} : \begin{cases} V \otimes W \to W \otimes V(x \otimes y) \\ x \otimes y \mapsto \tau(\mathbf{R} \cdot (x \otimes y)) \end{cases}$$

It is a map of $\mathcal{U}_q$-modules.

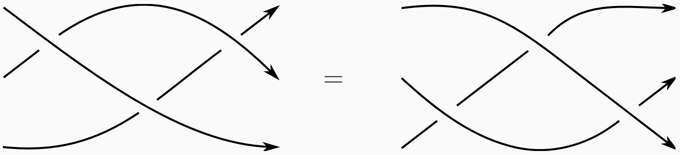
We can draw $c_{V,W}$ as a diagram:



which looks like a braid generator.

The braiding is a braiding

This diagram for $c_{-,-}$ is justified by the braid relation/RIII move:



Equivalent to the Yang-Baxter relation for $\mathbf{R}$.

Also, the braidings are always invertible, which gives us the RII move.

The RT functor for braids

Theorem (The Reshetikhin-Turaev functor)

There is a functor

$$\mathcal{F} : \mathrm{CBraid}_{\mathcal{U}_q} \to \mathcal{U}_q\text{-Mod}$$

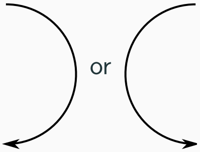
where $\mathrm{CBraid}_{\mathcal{U}_q}$ *is the category of braids with components labeled (colored) by objects of* $\mathcal{U}_q$-Mod.

Idea.

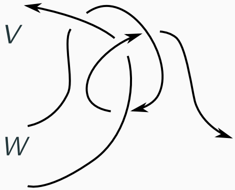
Strands labeled by V go to V, braidings go to the braiding. Because $c_{-,-}$ satisfy braid relations this is well-defined. □

The RT functor for tangles

More generally we can define $\mathcal{F}$ on oriented tangles, which in addition to braided parts can look like



For example, the image of



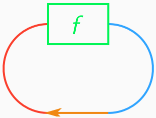
under $\mathcal{F}$ is a map $V^* \otimes W \otimes V \to W$.

Evaluation, coevaluation, and trace

Let V be a vector space with basis $\{v_i\}$ and dual basis $\{v^i\}$ of V^*. We can compute the trace of a linear map $f : V \to V$ with matrix elements ${f_i}^j$ by

$$1 \xmapsto{\mathrm{coev}_V} \sum_i v_i \otimes v^i \xmapsto{f \otimes \mathrm{id}_{V^*}} \sum_{ij} {f_i}^j v_j \otimes v^i \xmapsto{\mathrm{ev}_V} \sum_{ij} {f_i}^j v^i(v_j)$$
$$= \sum_i {f_i}^i = \mathrm{tr}\, f\,.$$

Diagrammatically:



Quantum trace

Orientations matter! Really there are two evaluations/coevaluations:

$$\mathrm{coev}_V^{\downarrow} : \mathbb{k} \to V^* \otimes V$$
$$\mathrm{coev}_V^{\downarrow}(1) = \sum_i v^i \otimes v_i$$

$$\mathrm{ev}_V^{\downarrow} : V \otimes V^* \to \mathbb{k}$$
$$\mathrm{ev}_V^{\downarrow}(v \otimes f) = f(\varpi^{-1} \cdot v)$$

$$\mathrm{coev}_V^{\uparrow} : \mathbb{k} \to V \otimes V^*$$
$$\mathrm{coev}_V^{\uparrow}(1) = \sum_i v^i \otimes \varpi \cdot v_i$$

$$\mathrm{ev}_V^{\uparrow} : V^* \otimes V \to \mathbb{k}$$
$$\mathrm{ev}_V^{\uparrow}(v \otimes f) = f(\varpi^{-1} \cdot v)$$

ϖ is the pivotal element, in our examples a power of K.

Quantum dimension

Definition

The quantum dimension of a $\mathcal{U}_q$-module V is

$$\dim_q(V) = \mathrm{tr}_q(\mathrm{id}_V) = \mathrm{ev}_V^{\downarrow}(\mathrm{id}_V \otimes \mathrm{id}_{V^*})\,\mathrm{coev}_V^{\uparrow}$$

Example

For the N-dimensional irrep V_{N-1} of $\mathcal{U}_q$,

$$\dim_q(V_{N-1}) = [N]_q = \frac{q^N - q^{-N}}{q - q^{-1}} = q^{N-1} + q^{N-3} + \cdots + q^{-(N-1)}$$

is a q-analog of N.

Quantum dimensions and semisimplicity

Recall a category is semisimple if objects are completely reducible (break apart into direct sums of simples).

General principle

$\mathcal{U}_q$-Mod is semisimple exactly when all the quantum dimensions of simple objects are nonzero.

To be precise, need to specify exactly what kind of category we are talking about. There are audience members who know more than me!

Link invariants from RT

Pick an object V of $\mathcal{U}_q$-Mod.

- A link L (with all components labeled by V) is a colored tangle diagram with no ends
- Its image under $\mathcal{F}$ is a linear map $\mathcal{F}_V(L) : \mathbb{C}(q) \to \mathbb{C}(q)$, which is a scalar.
- This scalar is an invariant of L.[1]
- Concretely, can compute

$$\mathcal{F}_V(L) = \mathrm{tr}_q\, \mathcal{F}_V(\beta)$$

where β is a braid whose closure is L.

[1] Technically depends on framing of L. Can get rid of this by normalizing braidings.

Examples of quantum link invariants

Jones polynomial

If $V = V_2$ is the 2-dimensional irrep of $\mathcal{U}_q$, get the Jones polynomial $\mathrm{V}_{2,L}$, a Laurent polynomial in q^2.

Colored Jones polynomial

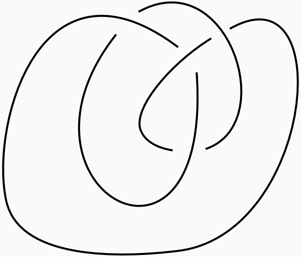
If $V = V_N$ is the N-dimensional irrep of $\mathcal{U}_q$, get the colored Jones polynomial $\mathrm{V}_{N,L}$, a Laurent polynomial in q^2.

HOMFLY-PT polynomial

If V is the N-dimensional irrep of $\mathcal{U}_q(\mathfrak{sl}_N)$, get the HOMFLY-PT polynomial, a Laurent polynomial in q^2 and $z = q^N - q^{-N}$.

More specific examples

The figure eight knot 4_1



has $\mathrm{V}_{2,4_1} = q^4 - q^2 + 1 - q^{-2} + q^{-4}$.
(Here we have normalized so $\mathrm{V}_{2,\text{unknot}} = 1$.)

More specific examples

Colored Jones at a root of unity

Set $\xi = \exp(\pi i/N)$ and $\{k\} = \xi^k - \xi^{-k}$. Then

$$\mathrm{V}_{N,4_1}(q = \xi) = \sum_{j=0}^{N-1} \prod_{k=1}^{j} \{N-k\}\{N+k\}.$$

- Computing these closed formulas for all N is hard!
- One reason: if K is presented as the closure of a braid on b strands, then computing $\mathrm{V}_{N,K}$ involves the trace of a $N^b \times N^b$ matrix.
- This one comes from writing 4_1 as surgery on the Borromean rings.

Geometric connections

Theorem

$$2\pi \lim_{N\to\infty} \frac{\log |\mathrm{V}_{N,4_1}(q=\xi)|}{N} = 2.02988\ldots = \mathrm{Vol}(4_1)$$

where $\mathrm{Vol}(K)$ *is the volume of the complete hyperbolic structure of* $S^3 \setminus K$.

Reminders:

- A hyperbolic knot has a complete finite-volume hyperbolic structure (metric of curvature -1) on its complement. This metric is a topological invariant.
- All knots are satellites, torus knots, or hyperbolic.

The volume conjecture

Conjecture ([Kas97; MM01])

For any hyperbolic knot K,

$$2\pi \lim_{N\to\infty} \frac{\log|\mathrm{V}_{N,K}(q=\xi)|}{N} = \mathrm{Vol}(K).$$

- There are versions for complex volume, for knots in 3-manifolds, for 3-manifolds...
- In every case where the left-hand limit is known to exist the conjecture holds.

How does V_N know about hyperbolic geometry?

Still not clear, but suggests we should study $\mathcal{U}_\xi$-Mod for $q = \xi$ more carefully.

Quantum groups at a root of unity

Quantum groups and quantum invariants

Quantum groups at a root of unity

Geometric applications

More complicated representation theory

Unlike for generic q, $\mathcal{U}_\xi$-Mod for $q = \xi = \exp(\pi i/N)$ is much more complicated:

- It is no longer semisimple
- There are uncountably many simple objects

What happens to the V_N?

Notice

$$\dim_q V_{n-1} = [n]_\xi = \frac{\xi^n - \xi^{-n}}{\xi - \xi^{-1}}$$

is zero for $n = N$, so $\dim_q V_N = 0$. We separate out:

$$\underbrace{V_1, V_2, \dots, V_{N-2},}_{\dim_q \neq 0} \quad \overbrace{V_{N-1}, V_N, \dots}^{\text{non-semisimple}}$$

Semisimplification

Traditional option here is to kill the non-semisimple part.

- Can specialize to "nice" modules by taking small quantum group $K^N = \pm 1, E^N = F^N = 0$, then send every negligible morphism (f with $\operatorname{tr}_q f = 0$) to zero.
- For Lusztig form of quantum group, instead want to start with category of tilting modules.
- Can also get the same category by using Temperly-Lieb diagrams at $q = \xi$ and quotienting by the Jones-Wenzl projectors for $n \geq N$.
- Result is a semisimple category, in fact a fusion category.
- These fusion categories are the input to the Rehsetikhin-Turaev and Turaev-Viro TQFTs.

A different approach

- For the volume conjecture we want to understand V_{N-1}, which is sent to 0 under semisimplification.
- In addition, there are a whole $\mathrm{SL}_2(\mathbb{C})$-worth of modules like V_{N-1}, also sent to 0.
- Can think of these modules as orthogonal to the semisimple part.
- By using these we can get new, interesting quantum invariants.

The big center

For q not a root of unity, center of $\mathcal{U}_q$ is generated by the Casimir $\Omega = FE + qK + q^{-1}K^{-1}$.

- At $q = \xi$, there is now a large central subalgebra $\mathcal{Z}_0 = \mathbb{C}[K^{\pm N}, E^N, F^N]$.
- Full center is $\mathcal{Z} = \mathcal{Z}_0[\Omega]/$polynomial relation
- For simple module V, action of $\mathcal{U}_\xi$ factors through some $\mathcal{Z}$-character $\chi : \mathcal{Z} \to \mathbb{C}$.
- For now, let's focus on $\mathcal{Z}_0$-characters $\chi : \mathcal{Z}_0 \to \mathbb{C}$.

The algebraic group $\mathcal{Z}_0$

- $\mathcal{Z}_0$ is a commutative Hopf algebra, so its spectrum is an algebraic group
- Specifically, it's the group $\mathrm{SL}_2(\mathbb{C})^*$ of matrices of the form

$$\chi = (\chi^+, \chi^-) = \left(\begin{bmatrix} \chi(K^N) & 0 \\ \chi(K^N F^N) & 1 \end{bmatrix}, \begin{bmatrix} 1 & \chi(E^N) \\ 0 & \chi(K^N) \end{bmatrix} \right)$$

- We call $\mathrm{SL}_2(\mathbb{C})^*$ the Poisson dual group of $\mathrm{SL}_2(\mathbb{C})$
- Because any simple $\mathcal{U}_\xi$-module has a $\mathcal{Z}_0$-character, $\mathcal{U}_\xi$-Mod will be $\mathrm{Spec}(\mathcal{Z}_0) = \mathrm{SL}_2(\mathbb{C})^*$-graded.

$\mathrm{SL}_2(\mathbb{C})^*$-grading on modules

Proposition

$\mathcal{U}_\xi$-Mod $= \bigoplus_{\chi \in \mathrm{SL}_2(\mathbb{C})^*} \mathcal{U}_\xi$-Mod$_\chi$ is $\mathrm{SL}_2(\mathbb{C})^*$-graded. Each graded piece has finitely many simple objects.

Proof.

If V_1, V_2 have $\mathcal{Z}_0$-characters χ_1, χ_2, then $V_1 \otimes V_2$ has $\mathcal{Z}_0$-character $\chi_1\chi_2$: for any $z \in \mathcal{Z}_0$,

$$\begin{aligned} z \cdot (v_1 \otimes v_2) &= \Delta(z) \cdot (v_1 \otimes v_2) = \sum z_{(1)} \cdot v_1 \otimes z_{(2)} \cdot v_2 \\ &= \sum \chi_1(z_{(1)}) v_1 \otimes \chi_2(z_{(2)}) v_2 = (\chi_1 \otimes \chi_2)(\Delta(z)) v_1 \otimes v_2 \\ &= (\chi_1\chi_2)(z)(v_1 \otimes v_2) \end{aligned}$$

□

Role of the Casimir

We think of the matrix for χ as corresponding to

$$\psi(\chi) = \chi^+(\chi^-)^{-1} = \begin{bmatrix} \chi(K^N) & -\chi(E^N) \\ \chi(K^N F^N) & \chi(K^N) - \chi(K^N E^N F^N) \end{bmatrix} \in \mathsf{SL}_2(\mathbb{C})$$

- Action of central Casimir Ω given by Nth root of an eigenvalue of $\psi(\chi)$
- Characters $\widehat{\chi} : \mathcal{Z} \to \mathbb{C}$ are in bijection with simple $\mathcal{U}_\xi$-modules.
- In particular, for any[2] $\chi : \mathcal{Z}_0 \to \mathbb{C}$ there are N corresponding $\mathcal{Z}$-characters and N irreps with $\mathcal{Z}_0$-character χ

[2] Unless $\chi = \pm 1$

Examples of modules

- Suppose $\chi = \pm\,\mathsf{id} \in \mathsf{SL}_2(\mathbb{C})^*$, so $\chi(K^N) = \pm 1, \chi(E^N), \chi(F^N)$.
- Modules with character $\pm\,\mathsf{id}$ are exactly modules for the small quantum group $\mathcal{U}_\xi/[K^{\pm 2N} - 1, E^N, F^N]$.
- These include all the usual $\mathcal{U}_q$-modules with integral highest weights
- The usual N-dimensional irrep V_{N-1} at $q = \xi$ has character $(-1)^{N+1}\,\mathsf{id}$ and gives $\mathrm{V}_{N,L}(q = \xi)$

Generic-highest-weight modules

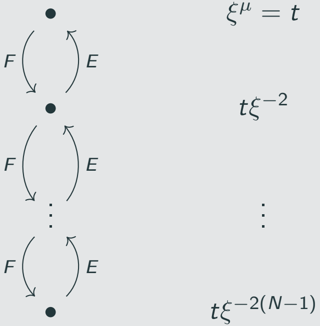
- Even if E^N and F^N act by 0 there are N-dimensional modules with non-integral highest weight.
- Say

$$\chi = \left(\begin{bmatrix} t & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & t \end{bmatrix} \right)$$

  i.e. $\chi(K) = t^N, \chi(E^N) = \chi(F^N) = 0$.
- Here $t = \xi^\mu \in \mathbb{C} \setminus \{0\}$ is like a multiplicative highest weight

Diagram

Modules V_μ with character χ look like



Non-semisimple invariants

Definition

Invariant coming from V_μ is the Nth ADO invariant (or colored Alexander polynomial) [ADO92].

- Can apply usual RT construction to the modules V_μ; since E and F act nilpotently, **R** converges
- Requires choice of $\mu \in \mathbb{C}/2N\mathbb{Z}, \mu \neq 0, \dots, N-2$ for each link component
- Because μ is generic, can think of ADO invariant as a Laurent polynomial in $t = \xi^\mu$.
- For $N = 2$, get the Alexander polynomial (specifically, Conway potential)
- Value at $\mu = N - 1$ is $\mathrm{V}_{N,L}(q = \xi)$, as in volume conjecture

Non-semisimple TQFT

Theorem (Blanchet, Costantino, Geer, and Patureau-Mirand [Bla+16])

These invariants extend to a TQFT for each $N \geq 2$.

- Defined on category of manifolds with choice of class in $H^1(M; \mathbb{C}/2N\mathbb{Z})$ generalizing our choice of μs before
- For $N = 2$, get a normalized Reidemeister torsion/Alexander polynomial for manifolds
- Mapping class groups here appear to be more powerful than in WRT TQFT: some Dehn twists have infinite order, for example

Modified dimensions

One technical issue:

$$\dim_q V_\mu = \mathrm{tr}_q(\mathrm{id}_{V_\mu}) = \mathrm{tr}(K^{N-1}|_{V_\mu}) = \mu(1 + \xi^{-2} + \cdots + \xi^{-2N+2}) = 0$$

so naive RT gives uniformly zero invariants.
To fix, consider value on $1-1$ tangles instead:

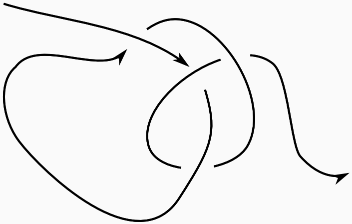


Image $\mathcal{F}(T)$ under functor is a map $V_\mu \to V_\mu$

Modified dimensions

- Since $\mathcal{F}(T)$ is an endo of a simple object, we have $\mathcal{F}(T) = \langle \mathcal{F}(T) \rangle \, \mathrm{id}_{V_\mu}$
- Can think of $\langle \mathcal{F}(T) \rangle$ as an invariant of closure L of T.
- How do we know that it doesn't depend on where we cut open the diagram of L?
- Not hard to show it doesn't matter if all components of L are colored by same V_μ. If not:

Theorem

There is a modified dimension function $d(V_\mu)$ such that

$$\mathrm{F}(L) = d(V_\mu) \, \langle \mathcal{F}(T) \rangle$$

is an invariant of L, where $T : V_\mu \to V_\mu$ is a $1-1$ tangle whose closure is L.

Computing modified dimensions

Essentially unique choice is

$$d(V_\mu) = \frac{\xi^{\mu} - \xi^{-\mu}}{\xi^{N\mu} - \xi^{-N\mu}}$$

- Akutsu, Deguchi, and Ohtsuki [ADO92] figured out the right $d(V_\mu)$
- General construction involving ratios of open Hopf links given by Geer, Patureau-Mirand, and Turaev [GPT09].
- Idea: quantum dimensions of all V_μ are 0. If we divide through by V_{N-1}, ratio gives something nonzero.

What about non-nilpotent modules?

- Our modules V_μ still had E and F act nilpotently
- For this reason the RT construction basically went through the same
- What happens at non-diagonal characters

$$\chi = \left(\begin{bmatrix} \kappa & 0 \\ \phi & 1 \end{bmatrix} \begin{bmatrix} 1 & \epsilon \\ 0 & \kappa \end{bmatrix} \right)?$$

- We already see that we get a category with nonabelian grading, because $\mathrm{SL}_2(\mathbb{C})^*$ is nonabelian

Geometric applications

Quantum groups and quantum invariants

Quantum groups at a root of unity

Geometric applications

The goal

- Previously we had invariants of links with abelian data (cohomology class).
- Now we will get invariants of links with *nonabelian* data
- We are working towards holonomy invariants:

A holonomy invariant

Theorem (Blanchet, Geer, Patureau-Mirand, and Reshetikhin [Bla+20])

There is a quantum invariant $\mathrm{F}_{N,L}(\rho)$ *of a link* L *plus an extended representation*[3]

$$\rho : \pi_1(S^3 \setminus K) \to \mathsf{SL}_2(\mathbb{C}).$$

$\mathrm{F}_{N,L}(\rho)$ *depends only on the conjugacy class of* ρ*. When* $\rho = (-1)^{N+1}$ *is trivial, we recover colored Jones* $\mathrm{V}_{N,L}(q = \xi)$ *at a root of unity.*

This is a nonabelian deformation of the Jones polynomial at a root of unity.

[3] More on this later

Significance

- Since $\mathrm{Isom}(\mathbb{H}^3) = \mathrm{PSL}_2(\mathbb{C})$, can describe hyperbolic structures on link complements with reps into $\mathrm{SL}_2(\mathbb{C})$.
- For each link L, $\mathrm{F}_{N,L}$ is a function on the extended character variety $\mathfrak{X}_N(L)$ of L.
- This is a simple generalization of the usual character variety $\mathfrak{X}(L)$, which is the moduli space of $\mathrm{SL}_2(\mathbb{C})$-reps of $\pi_1(S^3 \setminus L)$
- We can therefore put geometry in our quantum invariants
- Should be more powerful than ordinary quantum invariants

Application to the volume conjecture

Volume conjecture becomes:

1. Asymptotics of $\mathrm{F}_{N,K}(\rho_{\mathsf{hyp}})$ at complete hyperbolic structure ρ_{hyp} computes $\mathrm{Vol}(K)$
2. Asymptotics of $\mathrm{F}_{N,K}(\rho_{\mathsf{hyp}})$ and $\mathrm{F}_{N,K}((-1)^{N+1})$ at two points of $\mathfrak{X}_N(K)$ are related

Cyclic modules

Consider $\mathcal{Z}_0$-character

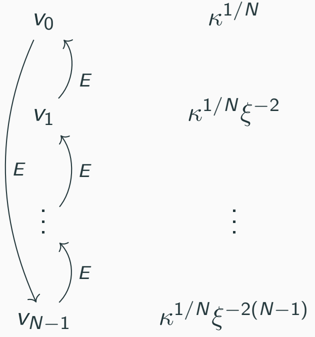
$$\chi = \left(\begin{bmatrix} \kappa & 0 \\ \phi & 1 \end{bmatrix} \begin{bmatrix} 1 & \epsilon \\ 0 & \kappa \end{bmatrix} \right)$$

Since $\chi(E^N), \chi(F^N)$ are nonzero, get a cyclic module $V_{\chi,\mu}$.
Here μ satisfies

$$-(\mu^N + \mu^{-N}) = \operatorname{tr} \psi(\chi) = \operatorname{tr} \chi^+(\chi^-)^{-1}$$

and gives action $\xi\mu + \xi^{-1}\mu^{-1}$ of Casimir.

$$\chi(K^N) = \kappa, \chi(E^N) = \epsilon$$



$$E \cdot v_k = v_{k-1}$$

$$E \cdot v_0 = \epsilon v_{N-1}$$

Braiding for cyclic modules

- Since $E^N, F^N \neq 0$, action of universal R-matrix $\mathbf{R}$ on cyclic modules does not converge.
- Instead consider automorphism

$$\mathcal{R} : \mathcal{U}_q \otimes \mathcal{U}_q \to \mathcal{U}_q \otimes \mathcal{U}_q$$

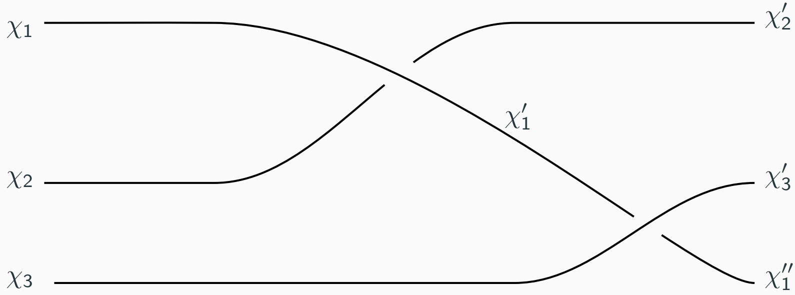
  given by $\mathcal{R}(x) = \mathbf{R} x \mathbf{R}^{-1}$.
- $\mathcal{R}$ still makes sense at $q = \xi$.
- Now a braiding is a map intertwining $\tau\mathcal{R}$:

$$c_{\chi_1,\chi_2} : V_{\chi_1} \otimes V_{\chi_2} \to V_{\chi_{2'}} \otimes V_{\chi_{1'}}$$
$$c(x \cdot v \otimes w) = (\tau\mathcal{R}(x)) \cdot c(v \otimes w)$$

- Because $\mathcal{R}$ acts nontrivially on $\mathcal{Z}_0 \otimes \mathcal{Z}_0$, characters on left and right are different!

Biquandles and characters

Leads to diagrams like:

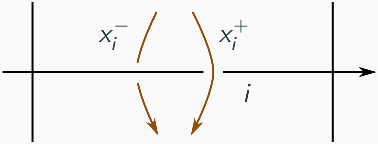


Notice *both* labels change at a crossing. What does this mean?

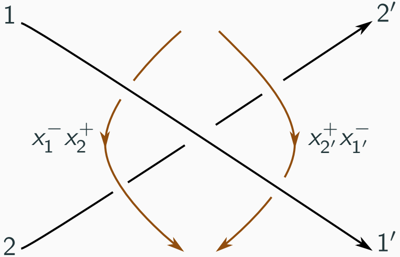
- Usual description (Wirtinger presentation) of knot group from a diagram has one generator for each arc.
- We instead want a *groupoid* with two generators for each segment.
- Path above a segment labeled by χ gives χ^+, path below gives χ^-
- Braiding on χ_i is a biquandle.

Braiding on characters

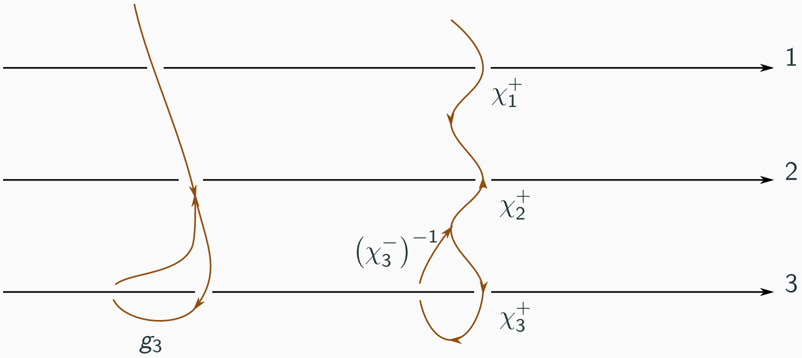
Can compute action of $\mathcal{R}$ on characters algebraically. Answer is better understood geometrically in terms of fundamental groupoid of tangle diagram:



The generators associated to segment i



There are relations at each crossing, such as the above



$$g_3 = \chi_1^+\chi_2^+\chi_3^+(\chi_3^-)^{-1}(\chi_2^+)^{-1}(\chi_1^+)^{-1}$$

Biquandle factorizations

Theorem ([Bla+20])

This is a generic biquandle factorization of the conjugation quandle of $\mathsf{SL}_2(\mathbb{C})$*:*

1. *Every* $\mathsf{SL}_2(\mathbb{C})^*$*-colored tangle has a well-defined rep* $\pi_1(\textit{complement}) \to \mathsf{SL}_2(\mathbb{C})$
2. *Not every rep* $\pi_1(\textit{complement} \to \mathsf{SL}_2(\mathbb{C})$ *can be expressed using* $\mathsf{SL}_2(\mathbb{C})^*$*-coords, but every rep is conjugate to one that can.*

How to define the invariant

Say we have a link L with $\rho : \pi_1(S^3 \setminus L) \to \mathrm{SL}_2(\mathbb{C})$.

1. Write L as a $1-1$ tangle diagram T.
2. Use factorization rule to color segments of T with $\chi \in \mathrm{SL}_2(\mathbb{C})^*$. (Might have to conjugate ρ first.)
3. Need extra data of Nth root μ of eigenvalues around each link component (to determine Casimirs)
4. Apply RT functor to get $\mathcal{F}_N(T) : V_{\chi,\mu} \to V_{\chi,\mu}$.
5. Then

$$\mathrm{F}_{N,L}(\rho) = d(V_{\chi,\mu}) \langle \mathcal{F}_N(T) \rangle$$

is our invariant.

Braiding for modules

- Still haven't quite defined the braiding. Condition

$$c(x \cdot v \otimes w) = (\tau\mathcal{R}(x)) \cdot c(v \otimes w)$$

  determines c up to a scalar.
- Determining the scalar is hard! Not even clear how to compute matrix coeffs of c
- Leads to phase ambiguities in definition of F_N
- With Reshetikhin, I am working on fixing these (probably requires extra structure on links).
- Preliminary computation of matrix coeffs of c is in my thesis [McP21a, Chapter 3]

Relation with the torsion

Because of this issue, very hard to actually compute F_N for nonabelian ρ. Currently working on more examples. One thing is known:

Theorem (Me [McP21b])

For any (L, ρ) with well-defined Reidemeister torsion $\tau_L(\rho)$,

$$\mathrm{F}_{2,L}(\rho)\mathrm{F}_{2,\overline{L}}(\overline{\rho}) = \tau_L(\rho).$$

This extends the definition of the Alexander polynomial as a quantum invariant from $\mathcal{U}_i$.

Proof strategy.

There is a Schur-Weyl duality between twisted Burau representation and action of $\mathcal{R}$ on $\mathcal{U}_i$. □

Thank you for watching!

Another holonomy invariant, with examples

Extending Kashaev and Reshetikhin [KR05], myself, Chen, Morrison, and Snyder [Che+21] constructed a holonomy invariant. Set $\zeta = \exp(2\pi i/\ell)$ for ℓ odd.

Fact

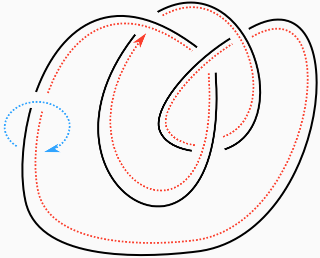
$\mathcal{U}_\zeta / \ker \chi$ is a simple bimodule of dimension N^2 for any $\mathcal{Z}$-character χ.

Theorem

By assigning a strand of a knot diagram with holonomy χ the module $\mathcal{U}_\zeta / \ker \chi$, we get a holonomy invariant $\mathrm{KR}_K(\rho)$ of knots. KR_K is a rational function on a N-fold cover $\mathfrak{X}_N(K)$ of $\mathfrak{X}(K)$.

For technical reasons it is much easier to define the braiding.

KR for the figure-eight knot



$K = 4_1$

longitude meridian

$$\mathfrak{X}(4_1) = \mathbb{C}[M^{\pm 1}, L^{\pm 1}]/\left\langle (L-1)(L^2M^4 + L(-M^8 + M^6 + 2M^4 + M^2 - 1) + M^4) \right\rangle$$

$M^{\pm 1}$ are the eigenvalues of the meridian and $L^{\pm 1}$ are the eigenvalues of the longitude.

To get $\mathfrak{X}_N(4_1)$, replace M with $\mu^N = M$

KR for the figure-eight knot

$(L-1)$ factor is the *commutative* component and the other is *geometric*. We compute that, for $N=3$,

$$\mathrm{KR}_K(\text{comm}) = \left(\mu^4 + 3\mu^2 + 5 + 3\mu^{-2} + \mu^{-4}\right)^2$$
$$\mathrm{KR}_K(\text{geom}) = 3(\mu^2+\mu^{-2})(\mu+1+\mu^{-1})^3(\mu-1+\mu^{-1})^3$$

Complete hyperbolic structure of 4_1 complement corresponds to points $\mu = 1, \exp(2\pi i/3), \exp(4\pi i/3)$ on geometric component.

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