Quantizing the Hyperbolic Volume

Links, etc. at
sl2-site

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Plan:

1. quantum invariants (Jones polynomial)
   state volume conjecture
2. complex volumes and how to compute them
3. 1 and 2 together $\rightsquigarrow$ new invariants V_N

hybrid btwn

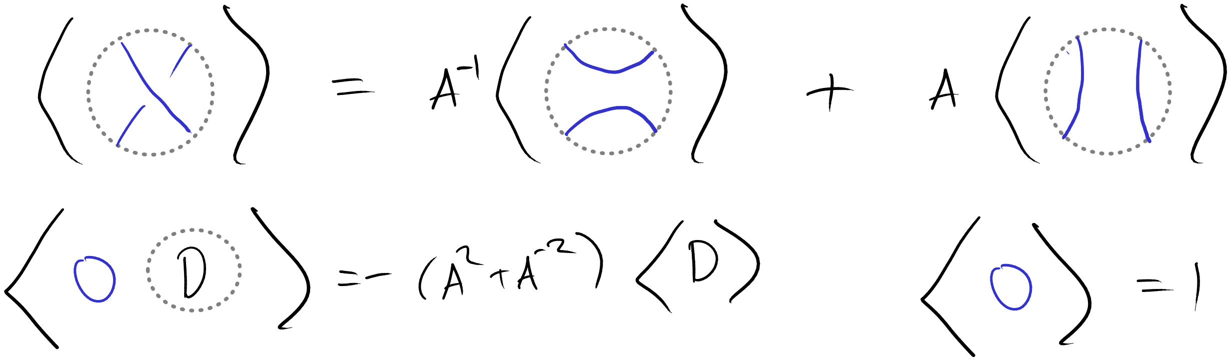
colored Jones polynomials

hyperbolic volume

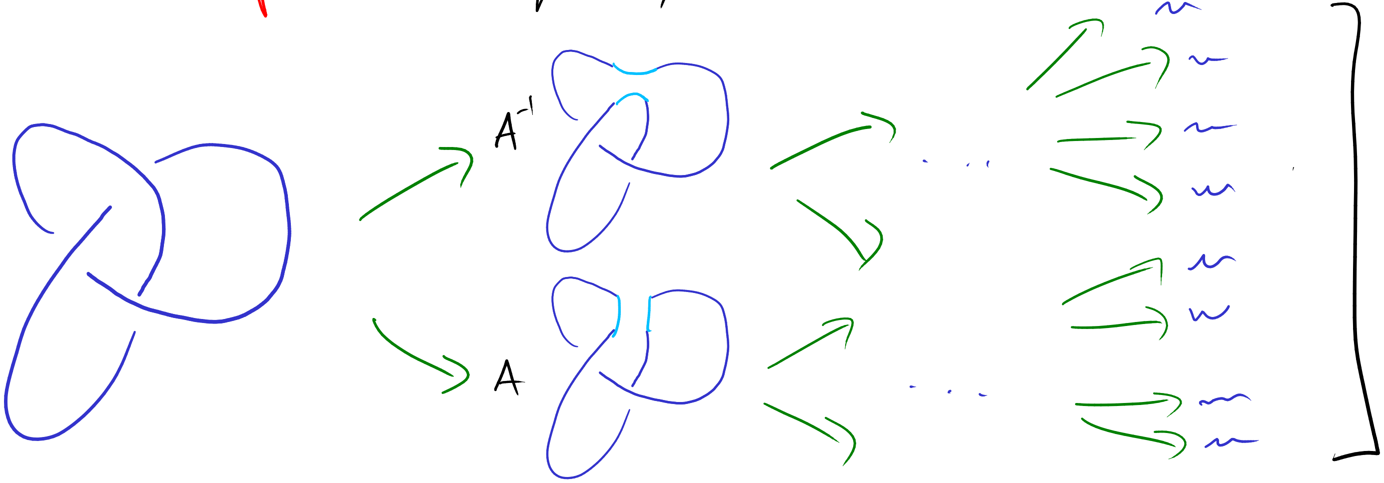
1. What is the Jones polynomial?

Kauffman bracket

One way: Pick a diagram D of L. Define $\langle D\rangle \in \mathbb{Z}[A,A^{-1}]$ by



Get tree of resolutions for D, ex



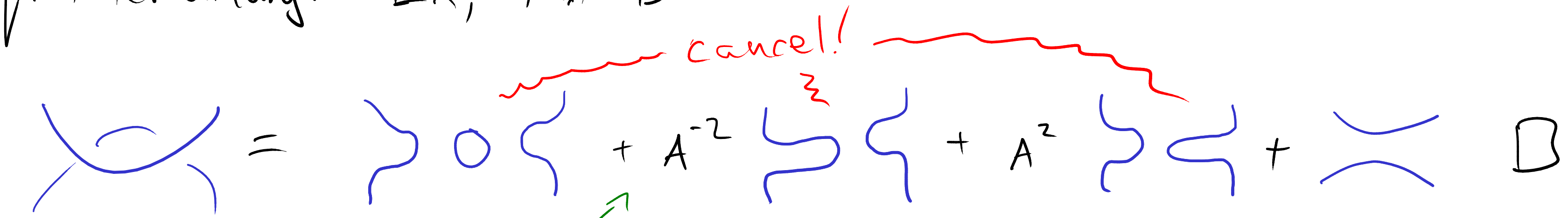
sum over diagrams of unlinks (power of $-A^2-A^{-2}$) times powers of A

In general, $2^{c(D)}$ terms

In this case $-A^{-5}-A^3+A^7$

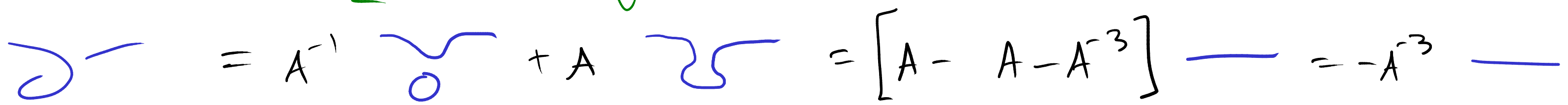
Theorem: $\langle D\rangle$ is an invariant of framed link L (no RI moves)

Pf: Elementary. Ex, RII is



Why no RI?

these are tangles, links w/ boundary



↑ writhe (winding # of blackboard
+1 normal vector field)

$$J_2(L) = \underbrace{(-A^{-3})^{-\overset{\text{writhe}}{w(D)}}}_{\text{"writhe correction"}} \langle D\rangle \in \mathbb{Z}[A, A^{-1}]$$ is the Jones polynomial.

Where does it come from? Jones: von Neumann algebras.
Different answer: representation theory.

V = vector space over $\mathbb{C}$

 $= V = \mathrm{id}_V$

 $= \mathrm{ev}_V : V \otimes V^* \to \mathbb{C}\,,\quad v \otimes f \longmapsto f(v)$

 $= \mathrm{id}_{V \otimes V}$

 $= \mathrm{coev}_V : \mathbb{C} \to V \otimes V^*,\quad 1 \longmapsto \sum_i \underset{\text{basis}}{v_i} \otimes \overset{\text{dual basis}}{v^i}$

 $= \mathrm{id}_{V^*}$

 $= \mathrm{ev} \circ \mathrm{coev} = \sum_i v^i(v_i) = \dim V$

More generally, these work for a monoidal category with duals (rigid, pivotal, spherical...)

Such things arise as H-mod for Hopf algebra H

Ex: $H = \mathbb{C}[G]$ group ring (G = group)

$H = \mathcal{U}(\mathfrak{g})$ universal enveloping algebra ($\mathfrak{g}$ = Lie alg)

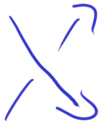
$H = \mathcal{U}_q(\mathfrak{g})$ quantum group, q-analogue of $\mathcal{U}(\mathfrak{g})$

$q = e^h$ quantization parameter

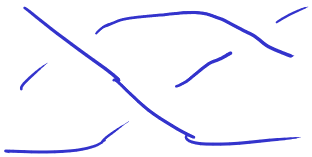
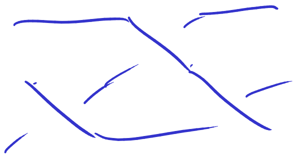
$A=\sqrt{-q}$

$\langle\ \rangle$ comes from $U_q(sl_2)$, $V=V_2=2$-dim'l simple module

 $= -A^2-A^{-2} = q+q^{-1} = \frac{q^2-q^{-2}}{q-q^{-1}} = [2]_q$ q-analogue of 2 quantum dimension

 is now a U_q-module morphism $c: V\otimes V \to V\otimes V$ braiding

R2 move says c is invertible, R3 is

 =  "braid relation"

$c\otimes id$ $\quad id\otimes c$ $\quad c\otimes id$ $\qquad id\otimes c$ $\quad c\otimes id$ $\quad id\otimes c$

$(c\otimes id)(id\otimes c)(c\otimes id) = (id\otimes c)(c\otimes id)(id\otimes c)$ "Yang-Baxter equation"

Where does c come from?

$U_q(sl_2)$ is quasitriangular*: universal R-matrix $\underline{R} \in U_q \otimes U_q$ so

$$c(x \otimes y) = \tau(\underline{R} \cdot (x \otimes y)) \qquad (\tau(v \otimes w) = w \otimes v)$$

is a U_q-module morphism and satisfies braid relations.

* technically, not quite: need $\hbar$-adic version w/ $q = e^{\hbar}$ b.c. $\underline{R}$ is a power series

Reshetikhin-Turaev construction: A ribbon† Hopf algebra like $U_q(g)$ gives link invariants for each module V

† quasitriangular + nice, compatible duals

Idea: Rules above turn a tangle diagram into an intertwiner: cups and caps are duals, crossings are $c^{\pm 1}$. For a link get a linear map $\mathbb{Z}[q,q^{-1}] \to \mathbb{Z}[q,q^{-1}]$, so an element of $\mathbb{Z}[q,q^{-1}]$.

Ex: V_2 = 2-diml irrep of $U_q(sl_2)$ gives Jones polynomial.

Ex: V_N = N-diml irrep of $U_q(sl_2)$ gives Nth colored Jones polynomial $J_N(L)$

Can compute diagrammatically as well: do < > but cable and project:

nth Jones-Wenzl projector (diagrammatic expression of projection $V_2^{\otimes N} \to V_N$)

What does $J_N(L)$, etc. mean?

Compare Alexander polynomial: lots of info about covers, fibering, etc.

One answer: quantum $SU(2)$ Chern–Simons theory. Hard to do mathematically.
due to Witten

More mathematical: Set $V_N(K) = \left.\dfrac{J_N(K)}{J_N(O)}\right|_{q=e^{2\pi i/N}}$ (Kashaev invariant).

Volume Conjecture [Kashaev, Murakami-Murakami]:

Let K be a hyperbolic knot. Then

$$\lim_{N\to\infty} \frac{\log V_N(K)}{N} = \frac{\mathrm{Vol}(K)}{2\pi}$$

← volume of complete hyperbolic metric on $S^3 \setminus K$

Why is this true??? Definition of J_N is very algebraic: how does geometry get involved?

There is a heuristic "proof" involving Gaussian integrals but it involves serious analytic difficulties, still unsolved except for special cases.

Recently me + N. Reshetikhin have a new link invariant we hope sheds light on this conjecture.

Theorem (to appear) For each $N \geq 2$ there is a quantized complex volume

$V_N(L, \rho, s) \in \mathbb{C}$

L = link in S^3

ρ = rep $\pi_1(S^3 \setminus L) \to SL_2(\mathbb{C})$

s = log-decoration (more on this later)

- conjugation-invariant
- like complex volume = hyperbolic volume + i Chern-Simons invariant
- $V_N(L, \rho = 1, s) = V_N(L)$ recovers Kashaev invariant

Conjecture: As $N \to \infty$,

appropriate lift of canonical structure to $SL_2(\mathbb{C})$

$$V_N(K, \rho, s) \sim (\text{constant})\, e^{\frac{N}{2\pi} \mathrm{Vol}_{\mathbb{C}}(K, \rho_{hyp}, s_0)}$$

← simple choice of log-decoration

for any hyperbolic K and ∂-parabolic ρ. $\rho = 1$ recovers KMM volume conjecture.

2. Complex volumes

hyperbolic manifold = locally modeled on $\mathbb{H}^3$

gives holonomy $\rho: \pi_1(M) \to \mathrm{Isom}(\mathbb{H}^3) = PSL_2(\mathbb{C})$

Def: Any $\rho: \pi_1(M) \to SL_2(\mathbb{C})$ is a generalized hyperbolic structure.
(Includes non-faithful, reducible, etc.)

$\rho_{hyp}: \pi_1(M) \to PSL_2(\mathbb{C})$ is unique (up to conjugacy) faithful rep for hyperbolic M.

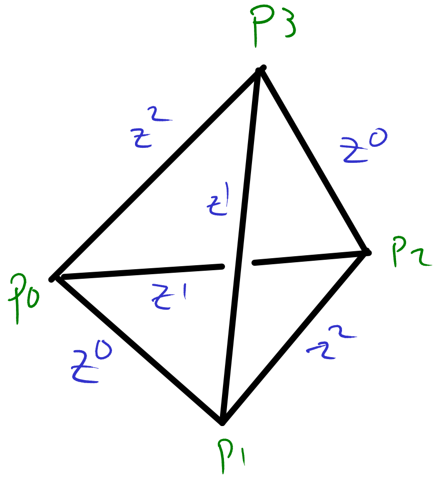
For $M = S^3 \setminus L$, can always lift to $SL_2(\mathbb{C})$ (choose a sign for each meridian)

Volume of ρ = volume of induced metric
(at least when ρ induces one)

Depends only on conjugacy class of ρ.

To compute in practice, especially when M has cusps:

- Ideally triangulate M
- Each tetrahedron has vertices in $\mathbb{CP}^1 = \partial \mathbb{H}^3$ coming from ρ



$z^0 = [p_0 : p_1 : p_2 : p_3]$ cross-ratio determines t up to congruence

z^0 = complexified dihedral angle

$z^1 = \frac{1}{1-z^0}$, $z^2 = \frac{1}{1-z^1} = 1 - \frac{1}{z^0}$ for other edges

Volume of one tetrahedron: $D(z^0) = D(z^1) = D(z^2)$ for

$D(z) = \mathrm{Im}(\mathrm{Li}_2(z)) + \arg(1-z)\log|z|$, $\mathrm{Li}_2(z) = \int_0^z -\log(1-u)\,\frac{du}{u}$ dilogarithm

$\mathrm{Vol}(\rho) = \sum_i D(z_i^0)$ sum over tetrahedra

Works for all ρ. If ρ is reducible/abelian, $\mathrm{Vol}(\rho) = 0$.

What about complex volume ?

$$\mathrm{Vol}_{\mathbb{C}}(\rho) = \mathrm{Vol}(\rho) + i\, CS(\rho) \in \mathbb{C}/4\pi^2 i\,\mathbb{Z}$$

Chern-Simons invariant of SO(3) frame field of metric

equiv:

$$\mathrm{Vol}_{\mathbb{C}}(\rho) = (\text{const}) \int_M tr\left(d\theta \wedge \theta + \tfrac{2}{3}\theta \wedge \theta \wedge \theta\right)$$

Chern-Simons form

θ = flat sl_2-connection w/ holonomy ρ

Now for tetrahedron, $\mathrm{Vol}_{\mathbb{C}} = -i\,\mathcal{L}(\zeta^0, \zeta^1)$, where

$$\mathcal{L}(\zeta^0, \zeta^1) = Li_2(e^{2\pi i \zeta^0}) + \frac{(2\pi i^2)}{2}\zeta^0 \zeta^1 + 2\pi i \zeta^0 \log(1 - e^{2\pi i \zeta^0})$$

for flattening $e^{2\pi i \zeta^0} = z^0$, $e^{2\pi i \zeta} = \dfrac{1}{1-z^0} = z^1$

<u>Extra data</u>: represent boundary conditions for flat connection θ.

For whole manifold M, need to make these coherently to get a flattening of M. If you do:

Theorem (Neumann) For closed (M, ρ) sum over flattened tetrahedra gives well-defined complex volume.

open regular neighborhood

For links, most natural to consider exterior $E(L) = S^3 \setminus \nu(L)$

$E(L)$ is a compact manifold with torus boundary

Now flattening along torus boundary is extra data, need to choose. This is the log-decorations mentioned before.

Theorem (me, Goerner-Zickert) If you choose a lift of ρ to $SL_2(\mathbb{C})$ and a log-decoration s of ρ,

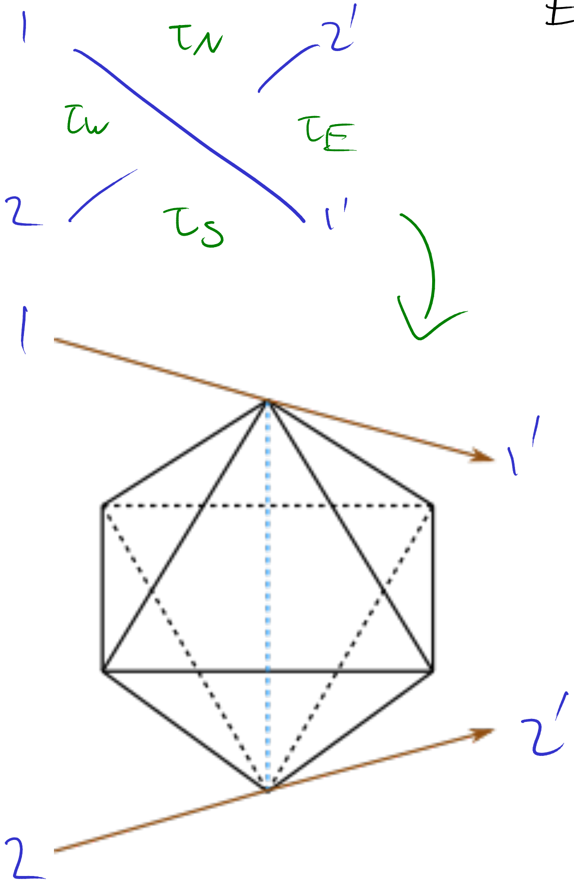
$\mathrm{Vol}_{\mathbb{C}}(E(L), \rho, s) \in \mathbb{C}/4\pi^2 i \mathbb{Z}$ is well-defined and computable by dilogarithm sum

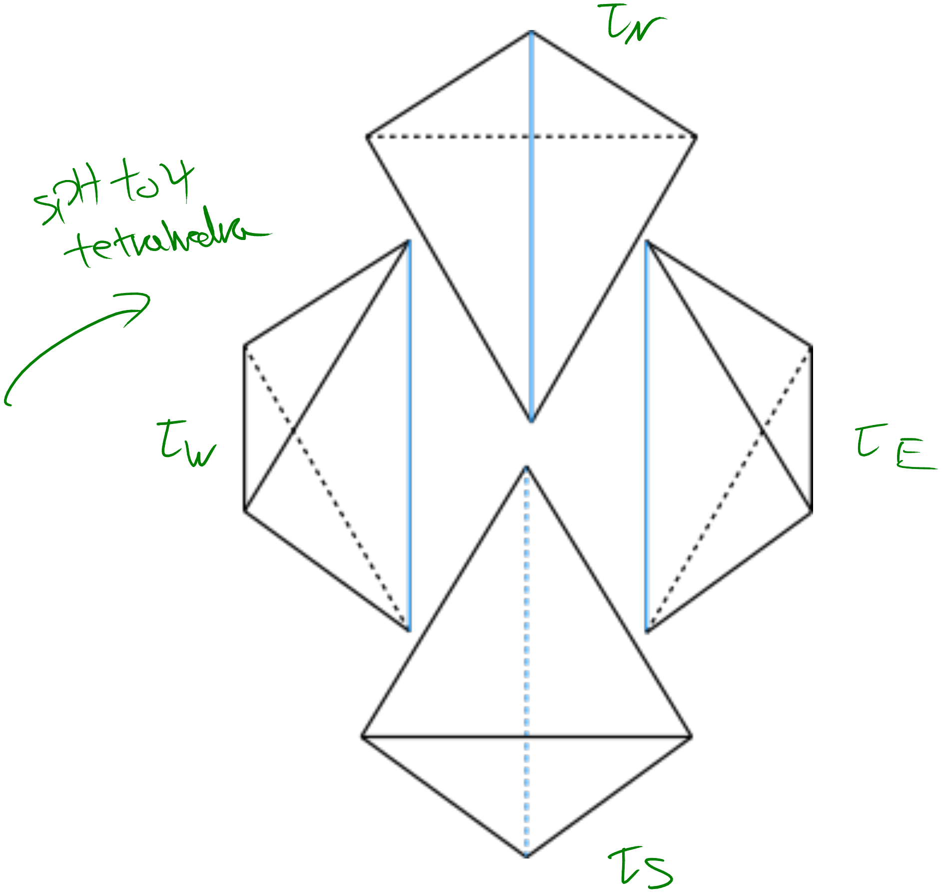
Explicit formula for dependence on s, which is simple.

For link diagrams can use octahedral decomposition (D. Thurston, Kashaev)

We follow Kim, Kim, Yoon arXiv:1612.02928

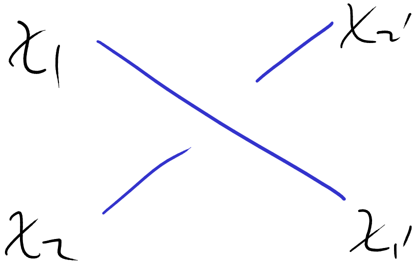
Each crossing gets an ideal octahedron!





see arXiv:2203.06042 for details. There called "shaping"

Shapes come from χ-colorings of diagram segments



$\chi_i = (a_i, b_i, m_i) \in (\mathbb{C} \setminus \{0\})^3$

relations at each crossing

m_i = meridian eigenvalues, ∂-parabolic means $m_i = 1$

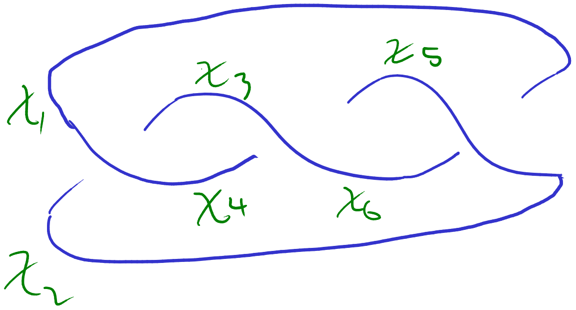
shape of each tetrahedron = ratio of adjacent b_i
(with factors of m_i)

a, b, m are deformed Ptolemy coordinates so logarithms give flattening.

log decoration = logarithms of meridian + longitude eigenvalues
(log-lift of holonomy of peripheral subgroups)

related to $\log(m_i)$ and sums of $\log(b_i)$

Ex: To find reps ρ of trefoil complement, find $\chi_1, \dots, \chi_6$



satisfying some relations at each crossing.

Theorem: For any diagram D of L and any $\rho : \pi_1(E(L)) \to SL_2(\mathbb{C})$, might not be a χ-coloring with holonomy ρ.

But there is a conjugate $\rho' = g \rho g^{-1}$ for which there is a χ-coloring.

Summary:

diagram

χ-coloring

logarithms determining a flattening

$$\mathrm{Vol}_{\mathbb{C}}(L, \rho, s) = \mathrm{Vol}_{\mathbb{C}}(D, \chi, f)$$

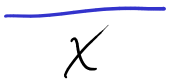
$$= \sum_{\text{crossings}} \sum_{\substack{\text{tetrahedra } i \\ \text{at crossing}}} \mathcal{L}(\xi_i^0, \xi_i^1)$$

log-parameters determined by f

Since we can compute from a diagram, this looks kind of like Reshetikhin-Turaev.

3. Our result is that it is! To define $V_N(L,\rho,s)$:

- choose a flattened diagram (D,χ,f) representing (L,ρ,s)
- strand

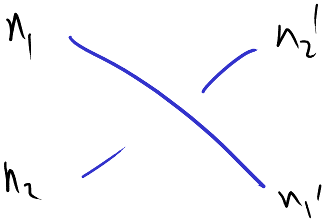


represents irreducible $\mathcal{U}_\xi(\mathfrak{sl}_2)$-module $V(\chi)$ of dimension N, central character χ

↖ $\xi = \exp(\pi i/N)$

The $V(\chi)$ are quite unusual and require q a root of unity
For example, E could act non-nilpotently!

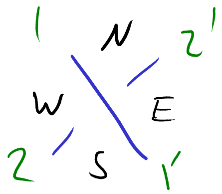
Braiding is a tensor $V(\chi_1)\otimes V(\chi_2) \to V(\chi_2')\otimes V(\chi_1')$



$$v_{n_1}\otimes v_{n_2} \longmapsto \sum_{n_1' n_2'} R^{n_1' n_2'}_{n_1 n_2}\, v_{n_2'}\otimes v_{n_1'}$$

key part is tensor $R = R(\chi, f)$ depending on geom. data

Recall labeling:



We show that

$$R^{n_1' n_2'}_{n_1 n_2} =_{(\text{normalization})} \frac{\Lambda(\zeta_N^0, \zeta_N^1 \mid n_2' - n_1)\, \Lambda(\zeta_S^0, \zeta_S^1 \mid n_2 - n_1')}{\Lambda(\zeta_W^0, \zeta_W^1 \mid n_2' - n_1' - 1)\, \Lambda(\zeta_E^0, \zeta_E^1 \mid n_2' - n_1')}$$

leads* to the link invariant $V_N(L, \rho, s)$

* some technicalities w/ modified dimensions as usual for non-semisimple invariants

Here $\Lambda(\zeta^0, \zeta^1 \mid n)$ is a quantum dilogarithm

Behaves sort of like a q-factorial:

$$\Lambda(\zeta^0, \zeta^1 \mid n) = \Lambda(\zeta^0, \zeta^1 \mid 0)\ \omega^{n \zeta^1} (1 - \omega^{\zeta^0 + 1})^{-1} \cdots (1 - \omega^{\zeta^0 + n})^{-1}, \quad \omega = e^{2\pi i / N}$$

Value $\Lambda(\zeta^0, \zeta^1 \mid 0)$ is related to $\exp(\mathcal{L}(\zeta^0, \zeta^1) / 2\pi i N)$.

Also,

$$\Lambda(\zeta^0, \zeta^1 \mid 0) \sim \exp^{\frac{N}{2\pi i} \mathrm{Li}_2(e^{2\pi i \zeta^0})} \quad \text{as } N \to \infty$$

Λ defined using Faddeev's noncompact quantum dilogarithm.

Proof idea: Relies on "Kashaev-Reshetikhin construction", a variant of Reshetikhin-Turaev.

Can no longer directly use $\underline{R}$ to find R (it might not converge if $E^N, F^N \neq 0$)

Workaround determines $R^{n_1' n_2'}_{n_1 n_2}$ projectively.

Formula above (4 q-factorials, like Kashaev, but not Jones) comes from basis Fourier-dual to highest-weight basis.

Hard part: finding the right normalization. To do it, we analogy with $\mathrm{Vol}_{\mathbb{C}}$.

Proof of invariance uses largely the same combinatorial reasoning as $\mathrm{Vol}_{\mathbb{C}}$ for octahedral decomposition.

Properties of V_N:

- Defined for any (L,ρ,s) (up to a minor condition on s when ρ is boundary-parabolic)
- Simple dependence on choice of s similar to $e^{\mathrm{Vol}_{\mathbb{C}}/2\pi}$
- gauge-invariant (depends only on conjugacy class of ρ)
- When ρ is trivial, get Kashaev invariant
- When ρ is abelian, get ADO invariants
  Akutsu-Deguchi-Ohtsuki
- Can phrase as:

$V_N(L)$ is a function on (a log-decorated version of) the $SL_2(\mathbb{C})$-character variety of L

To justify the title of this talk:

☆ quantum (comes from quantum groups, have state spaces unlike classical case, related to quantum $Sl_2(\mathbb{C})$ Chern-Simons theory)

☆ complex volume (formula for braiding involves 4 dilogarithms just like $\mathrm{Vol}_{\mathbb{C}}$, same log-dec dependence)

If V_N quantizes $\mathrm{Vol}_{\mathbb{C}}$, might expect in semiclassical limit $N \to \infty$

$\hbar = \frac{2\pi i}{N}$

$$V_N(L, \rho, s) \sim e^{N \mathrm{Vol}_{\mathbb{C}}(L, \rho, s)/2\pi i}$$

we recover classical $\mathrm{Vol}_{\mathbb{C}}$ from quantum V_N.

Actually, we expect something stranger:

Conjecture: Let K be a hyperbolic knot, (ρ_{hyp}, s_0) a certain natural choice of log-decoration for it's faithful holonomy.

Then for <u>any</u> boundary parabolic ρ and a certain s,

↙one-loop invariant of Dimofte-Garoufalidis

$$V_N(K, \boxed{\rho}, \boxed{s}) \sim S_1(K, \rho_{hyp}, s_0)\, e^{N \,\mathrm{Vol}_{\mathbb{C}}(K, \rho_{hyp}, s_0)} \quad \text{as } N \to \infty$$

Surprising part: <u>any</u> ρ, not just ρ_{hyp}.

In particular, even geometrically trivial $\rho = 1$ works. This is the original volume conjecture.

Why should this be true? For large N,

hard analysis is hiding here

$$V_N(K, \rho, s) \sim \int_\Gamma e^{N\Phi(b_1,\dots,b_n)}\, db_1 \dots db_n$$

looks like an integral of holomorphic potential function Φ over space of shape params.

Critical points of Φ = solns. of gluing eqs of triangulation

Can deform Γ to go through these and do Gaussian saddle-point approximation.

When deforming Γ the original value of ρ doesn't matter.