Cyclic modules for quantum $\mathfrak{sl}_2$ and hyperbolic knot theory

Abstract: The quantum group $\mathcal{U}_q(\mathfrak{sl}_2)$ is a q-analogue of the universal enveloping algebra of the Lie algebra $\mathfrak{sl}_2$. For generic values of q its representation theory is closely analogous to the classical case, but when q is a root of unity there are new phenomena; for example, there are "cyclic" modules where the (typically nilpotent) generators E and F act invertibly. Such modules have a geometric structure first studied by Kac and de Concini in the late 80s. In this talk I will describe a way to parametrize the cyclic modules using cluster-type coordinates. I will then explain how these coordinates are naturally related to the hyperbolic geometry of knots and links, and (if there's time) to the Volume Conjecture. Parts of this talk are based on joint work with Nicolai Reshetikhin.

Plan

- (5 min) reminders on $\mathfrak{sl}_2$ and quantum groups
- (10 min) cyclic modules and $\mathrm{SL}_2(\mathbb{C})^*$
- (7 min) the Kashaev-Reshetikhin braiding
- (8 min) octahedral coordinates for cyclic modules
- (15 min) octahedral coordinates and hyperbolic knots
- (5 min) What does this have to do with the Volume Conjecture?

1 Reminders on standard cases

1.1 The classical case

$\mathfrak{sl}_2$ is the Lie algebra of traceless 2×2 matrices over $\mathbb{C}$

H, E, F with

Fundamental rep

Highest weight reps indexed by nonnegative integers, Casimir picks out this parameter

Everything is completely reducible, i.e. $\mathcal{U}(\mathfrak{sl}_2)$ is semisimple

1.2 Quantum $\mathfrak{sl}_2$

$K = q^{\frac{H}{2}}$ so $KE = qEK$ and $KF = q^{-1}FK$

This is a q-analogue.

Has a Hopf algebra structure, so category of modules has tensor products and duals (also true for $\mathfrak{sl}_2$)

Underlies lots of quantum knot invariants like the Jones polynomial. Key thing is the **universal $\boldsymbol{R}$-matrix** $\boldsymbol{R} \in \mathcal{U}_q(\mathfrak{sl}_2) \hat{\otimes} \mathcal{U}_q(\mathfrak{sl}_2)$. Maps

$$B_{V,W} = \text{act by } \boldsymbol{R} \text{ then flip}$$

satisfy braid relation because $\boldsymbol{R}$ satisfies Yang-Baxter equation.

1.3 Modules: generic case

We say q is **generic** if $q^{\frac{n}{2}} \neq 1$ for any $n \in \mathbb{Z}$ (if it's not a root of unity)

The representation theory of this algebra is the same as classical $\mathfrak{sl}_2$:

Theorem For generic q there are two simple $\mathcal{U}_q(\mathfrak{sl}_2)$-modules of dimension n that differ by a sign. If we consider the category of reps where K acts diagonalizably (to avoid silly issues with Jordan blocks) then it is semisimple.

If you use the n-dimensional module V_{n-1} to define a link invariant you get the nth **colored Jones polynomial**. The Casimir acts on it by

$$q^{\frac{n-1}{2}} + q^{-\frac{n-1}{2}}$$

1.4 Modules: root of unity, small quantum group

Fix an integer $N \geq 2$. Set $\omega = e^{2\pi i/N}$.

Now K^N, E^N, F^N are central!

- Have a big Hopf subalgebra $\mathcal{Z}_0$ generated by these.
- $\mathcal{U}_\omega(\mathfrak{sl}_2)$ is N^3-dimensional over $\mathcal{Z}_0$.
- Full center is generated by Casimir modulo a polynomial rel.

Small quantum group is the quotient by $K^{2N} - 1, E^N, F^N$, which essentially kills $\mathcal{Z}_0$

- Its representation theory is related to $\mathfrak{sl}_2$ in positive characteristic.
- Highly non-semisimple
- There are simple modules $V_0, \cdots, V_{N-2}$ of dimension $1, \cdots, N-1$ that behave in the standard way
- When you get to bigger dimensions you get strange things: there are indecomposable, non-irreducible modules
- If you make it semisimple (take quotient by modules with vanishing quantum dimension) you get the Verlinde categories used in Witten-Reshetikhin-Turaev. This kills everything except $V_0, \cdots, V_{N-2}$

We want to study the modules thrown out by taking the small quantum group. They are related to hyperbolic geometry!

2 Cyclic modules

2.1 The central algebra

$\mathcal{Z}_0$ is a commutative Hopf algebra, so algebra of functions on some group. The group is

$$\mathrm{SL}_2(\mathbb{C})^* = \left\{ \left(\begin{pmatrix} k & 0 \\ kf & 1 \end{pmatrix}, \begin{pmatrix} 1 & e \\ 0 & k \end{pmatrix} \;\middle|\; k \neq 0 \right) \right\} \subset \mathrm{GL}_2(\mathbb{C}) \times \mathrm{GL}_2(\mathbb{C})$$

the **Poisson dual group** of $\mathrm{SL}_2(\mathbb{C})$. (People also call it the Langlands dual.)

Identify $\mathcal{Z}_0$ characters with group elements by

$$\chi \mapsto \left(\begin{pmatrix} \chi(K^N) & 0 \\ \chi(K^N F^N) & 1 \end{pmatrix}, \begin{pmatrix} 1 & \chi(E^N) \\ 0 & \chi(K^N) \end{pmatrix} \;\middle|\; k \neq 0 \right)$$

and this is a group isomorphism because of equations like

$$\begin{pmatrix} 1 & e_1 \\ 0 & k_1 \end{pmatrix} \begin{pmatrix} 1 & e_2 \\ 0 & k_2 \end{pmatrix} = \begin{pmatrix} 1 & e_1 k_2 + e_2 \\ 0 & k_1 k_2 \end{pmatrix}$$

and

$$\Delta(E^N) = E^N \otimes K^N + 1 \otimes E^N$$
$$\Delta(K^N) = K^N \otimes K^N$$

2.2 Geometric picture

Let $\mathcal{C}$ be the category of finite-dimensional $\mathcal{U}_\omega(\mathfrak{sl}_2)$-modules where $\mathcal{Z}_0$ acts diagonalizably.

Representation category is $\mathrm{SL}_2(\mathbb{C})^*$-graded:

$$\mathcal{C}_{\chi_1} \otimes \mathcal{C}_{\chi_2} \subset \mathcal{C}_{\chi_1 \chi_2}$$

Convenient to take birational map

$$\mathrm{SL}_2(\mathbb{C})^* \to \mathrm{SL}_2(\mathbb{C})$$
$$\begin{pmatrix} k & 0 \\ kf & 1 \end{pmatrix}, \begin{pmatrix} 1 & e \\ 0 & k \end{pmatrix} \mapsto \begin{pmatrix} k & -e \\ kf & k^{-1} - kef \end{pmatrix}$$

which is *not* a group homomorphism (more later!) Makes $\mathcal{C}$ a category fibered over $\mathrm{SL}_2(\mathbb{C})$, but not in the usual sense of grading.

$\mathrm{SL}_2(\mathbb{C})$ conjugacy classes break up as:

Identity Complicated, not semisimple

Diagonalizable Semisimple, split into N classes by Casimir

Parabolics Similar to above

2.2a Example

Suppose χ corresponds to diagonal matrix

$$\begin{pmatrix} m & 0 \\ 0 & m^{-1} \end{pmatrix}$$

Then the Casimir acts by $\omega^{\mu+\frac{1}{2}} + \omega^{-\mu-\frac{1}{2}}$ with $\omega^{N\mu} = m$ and

$$K \cdot v_k = \omega^{\alpha+k}$$

$$v_0 \xrightarrow{E} v_1 \xrightarrow{E} v_2 \to \cdots \to v_{N-1}$$

Kind of similar, just weird that highest weights lie in $\mathbb{C}$ not $\frac{1}{2}\mathbb{Z}$.

Leads to ADO invariants, including Alexander polynomial (at $N = 2$) for $t = \mu$

2.2b Example
Now for general one have eigenvectors

$$K \cdot v_k = \omega^{\alpha+k}$$

where $e^{2\pi i\alpha} = \chi(K^N)$. E moves between these:

$$v_0 \xrightarrow{E} v_1 \xrightarrow{E} v_2 \to \cdots \to v_{N-1} \to \varepsilon v_0$$

Can work out that Casimir has

$$\Omega \cdot v_k = \left[\omega^{\mu+\frac{1}{2}} + \omega^{-(\mu+\frac{1}{2})}\right] v_k$$

where $e^{\pm 2\pi i\mu}$ are eigenvalues of matrix of χ

Note:
- There is no highest weight, because ker $E = 0$
- The eigenvalues of K are *not* directly related to eiegenvalues of the Casimir
- The eigenvalues of the Casimir are what determine the iso class
- There is no clear way to choose a canonical basis like a weight basis for a $\mathfrak{sl}_2$-module

3 The braiding
[Kashaev-Reshtikhin] showed there is a braiding on these modules, in the sense that there for generic χ_1, χ_2 there is an algebra map

$$\mathcal{R} : \mathcal{U}_\omega(\mathfrak{sl}_2) \otimes \mathcal{U}_\omega(\mathfrak{sl}_2) \to \mathcal{U}_\omega(\mathfrak{sl}_2) \otimes \mathcal{U}_\omega(\mathfrak{sl}_2)[W^{-1}]$$

satisfying braid relations/Yang-Baxter equation. (Actually it's only defined generically, hence the localization.) It descends to a family of maps

$$\mathcal{U}_\omega(\mathfrak{sl}_2)/(\ker \chi_1) \otimes \mathcal{U}_\omega(\mathfrak{sl}_2)/(\ker \chi_2) \\ \to \\ \mathcal{U}_\omega(\mathfrak{sl}_2)/(\ker \chi_4) \otimes \mathcal{U}_\omega(\mathfrak{sl}_2)/(\ker \chi_3)$$

obeying "colored" or "holonomy" braid relation. Related to certain coordinates of space of $\mathfrak{sl}_2$ flat connections on braid complement. Parameters of χ_i are determined by

$$(\chi_3 \otimes \chi_4)\mathcal{R} = \chi_1 \otimes \chi_2$$

3.0a Problem

Suppose you are trying to use these to define knot invariants.

- The key ingredient is a braiding (aka R-matrix).
- For ordinary $\mathcal{U}_q(\mathfrak{sl}_2)$ modules this is a single $N^2 \times N^2$ matrix (for the Nth colored Jones polynomial)
- Now it is a *family* of matrices over all pairs of characters
- Unlike before we do not get matrix coefficients for action on modules: need to solve some equations and choose a normalization
  - This is again a family of normalizations over $\mathrm{SL}_2(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C})$

The coordinates $\chi(K^N), \chi(E^N), \chi(F^N)$ are hard to work with

- these recurrences depend
  - on the choice of generators of $\mathcal{U}_\omega(\mathfrak{sl}_2)$ and
  - on the matrix coefficients of the action on $V(\chi, \mu)$.
- If you use the choices I gave above it's a huge mess.
  - For example, the fact that ε shows up all at once at the end is very annoying.
- Parameters of χ_3, χ_4 are complicated rational functions in χ_1, χ_2 whose meaning is unclear

Deeper explanation: $\mathrm{SL}_2(\mathbb{C})^*$ is not $\mathrm{SL}_2(\mathbb{C})$!

- for applications to hyperbolic quantum topology we really want to use $\mathrm{SL}_2(\mathbb{C})$ because $\mathrm{PSL}_2(\mathbb{C})$ is the isometry group of $\mathbb{H}^3$.

- we want a $\mathrm{SL}_2(\mathbb{C})$-crossed braiding on our module category and that does not quite work: instead have some more complicated structure involving a "biquandle".

4 New parametrization

Consider Weyl algebra $\mathcal{W}_q$ with $XY = qYX$ and an extra central Z. (Could also call this a quantum torus.)

Lemma There is a homomorphism $\varphi : \mathcal{W}_q \to \mathcal{U}_q(\mathfrak{sl}_2)$ given by

$$\varphi(K) = X \quad \varphi(E) = q^{\frac{1}{2}} Y(Z - X) \quad \varphi(F) = Y^{-1}(1 - Z^{-1}X^{-1})$$

This is closely related to the standard cluster presentation of $\mathcal{U}_q(\mathfrak{sl}_2)$ (goes back to Faddeev).

Now at $q = \omega$ center is X^N, Y^N, Z and we have modules

$$X \cdot v_k = \omega^{\alpha+k} \quad Y \cdot v_k = \omega^{\beta+k+1} \quad Z \cdot v_k = \omega^{\mu}$$
$$\chi(X^N) = \omega^{N\alpha} = a \quad \chi(Y^N) = \omega^{N\beta} = b \quad \chi(Z^N) = \omega^{N\mu} = m$$

4.1 Geometric interpretation for a point

Again can consider family of cyclic modules $V(\chi, \mu)$, now parametrized by central $\mathcal{W}$-characters, or in terms of $\mathrm{SL}_2(\mathbb{C})$

$$\begin{pmatrix} a & -(a-m)b \\ (a - m^{-1})b^{-1} & m + m^{-1} - a \end{pmatrix}$$

We can specify a matrix in $\mathrm{SL}_2(\mathbb{C})$ by an eigenvalue, an eigenspace, and the action on a generic vector. Here

- b is the coordinate in $\mathbb{C}P^1$ of the eigenspace
- m^{-1} is the eigenvalue on that space
- a is the action on another vector

This is essentially the data of a pinned, framed flat $\mathfrak{sl}_2$ connection on a once-punctured disc [Goncharov-Shen]

4.2 Geometric interpretation for a crossing

At a crossing have four characters.

Again there are some (less) terrible rational functions for these.

Unlike before there is a natural geometric interpretation in terms of ideal triangulations!

4.3 Takeaway

Theorem [Me, Reshetikhin] For this family of modules

1. there are holonomy R-matrices with explicit formulas
2. They satisfy a suitable braid relation exactly, with no phase ambiguity (as existed in earlier solutions).

Key step for 1 is that Weyl generators give nice, straightforward recurrences that can be solved using Faddeev's quantum dilogarithm.

To understand 2 need geometric interpretation.

5 Octahedral coordinates

We call

- a labeling of a tangle diagram by parameters $\chi = (a, b, m)$
- satisfying the relation at each crossing induced by the KR braiding

an **octahedral coloring** of the tangle diagram. Why?

5.1 Ideal triangulations

Following Thurston, study hyperbolic structures on link complements by ideally triangulating them

- think of $\mathbb{H}^3$ as upper half-space
- boundary at ∞ is $\mathbb{C}P^1$ (with Euclidean metric)
- shows that $\operatorname{Isom} \mathbb{H}^3 = \mathrm{PSL}_2(\mathbb{C})$ acting by linear fractional transformations on boundary
- ideal tetrahedrahedon has vertices there

To get a link complement triangulate with vertices on link. Then **holonomy** of hyperbolic structure $\rho : S^3 \setminus L \to \mathrm{PSL}_2(\mathbb{C})$ (which up to conjugacy determines it) is given by, for example, LFTs that glue together along faces.

5.2 Shape parameters

In practice we don't (just) need matrices of ρ.

Geometry of ideal tetrahedron (with four vertices in $\mathbb{C}P^1$) is determined by cross-ratio. Called **shape parameter**.

- Natural geometric coordinates: can use them to find volumes, etc.

In practice we use **Thurston's gluing equations** in the shape parameters to solve for ρ.

5.3 Octahedral decompositions

For quantum topology want to work in terms of link diagrams.

A diagram D of L determines a (semi-)ideal triangulation of $S^3 \setminus L$ called the **octahedral decomposition**.

- four tetrahedra for each crossing (in the corners)

Instead of using shape parameters directly, use **octahedral coordinates** on each segment.

- The shape parameter of each tetrahedron is a ratio of the segment parameters
- Gluing equations become relations for each segment and region
- these are related to **Ptolemy coordinates** [Zickert] that greatly simplify the computation of the Chern-Simons invariant (good algebraic properties!)

Theorem [me] The KR braiding relations on the Weyl characters are equivalent to the octahedral equations. Extending some earlier work of [Cho] gives an explicit formula for them in terms of Wirtinger generators.

5.4 Consequences

Theorem The representation category is not only $\mathrm{SL}_2(\mathbb{C})^*$-graded, but can also be given a (generalization of a) $\mathrm{SL}_2(\mathbb{C})$-grading in a natural way compatible with tensor product

Theorem The quantum invariant constructed from the holonomy R-matrices is well-defined (not up to a phase, as in many earlier constructions of similar things) and is closely analogous to the Chern-Simons invariant (a quantization of it) hence to the hyperbolic volume.

6 The Volume Conjecture

Let X be the moduli space of hyperbolic structures on $S^3 \setminus L$, ie a version of the $\mathrm{SL}_2(\mathbb{C})$ character variety of L.

If L is hyperbolic X has a distinguished point ρ_{hyp}corresponding to the complete structure.

The Chern-Simons invariant is a section of a line bundle over X whose real part is e^{vol}.

Our quantum invariant is a section of a related vector bundle.

The Volume Conjecture says the large N behavior of this section at the trivial point of X is given by the value at ρ_{hyp}

Perhaps this invariant can reveal more connections explaining why this is true.