$SL_2(\mathbb{C})$-holonomy invariants of links

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Acknowledgements

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- and also to thank Carmen Caprau and Christine Ruey Shan Lee for organizing the session on quantum invariants and inviting me to speak.
- Much of the mathematics I will present is due to Kashaev-Reshetikhin and Blanchet, Geer, Patureau-Mirand, and Reshetikhin, although I will also discuss some of my own work (mostly in this part.)

Introduction

Overview

Part I (previously) General idea of and motivation for a *holonomy invarant* of a link L with a representation $\pi_1(S^3 \setminus L) \to G$.

Part II (now) Construction of a holonomy invariant for $G = \mathrm{SL}_2(\mathbb{C})$ due to Blanchet, Geer, Patureau-Mirand, and Reshetikhin [Bla+20]

- I will also discuss my recent work [McP20] interpreting their invariant in terms of the $\mathrm{SL}_2(\mathbb{C})$-twisted Reidemeister torsion.
- The plan is:
  1. Discuss some properties of the BGPR construction and how it relates to other link invariants
  2. Give an overview of the technical aspects of the construction

What is the BGPR invariant?

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BGPR invariant

- L a link in S^3 and ρ a representation $\pi_1(S^3 \setminus L) \to \mathrm{SL}_2(\mathbb{C})$.
- Pick an integer $r \geq 2$.
- Pick some rth roots: Let x_i be a meridian of the ith component of L such that $\rho(x_i)$ has eigenvalues $\lambda_i^{\pm}$. Choose rth roots $\mu_i^r = \lambda_i$.

The rth *BGPR* invariant is a complex number

$$F_r(L, \rho, \{\mu_i\})$$

defined up to an overall r^2th root of 1. Furthermore, F_r is invariant under global conjugation of ρ (i.e. it is *gauge invariant.*)

Caveat

F is currently only defined for $\lambda_i \neq \pm 1$. A fix is in preparation.

Abelian case

Here's a simple case:

- For any link L, pick $t \neq 0$. Then there is a representation sending every meridian x to

$$\rho(x) = \begin{pmatrix} t & \\ & t^{-1} \end{pmatrix}$$

- Any ρ with abelian image (which avoids ± 1 as eigenvalues) is conjugate to one of this type, but maybe different t_i for each component.
- In this special case, $F_r(L, \rho, \{t_i^{1/r}\})$ is equal to the rth *Akutsu-Deguchi-Ohtsuki* (ADO) invariant.
- For $r = 2$, $F_2(L, \rho, \{\sqrt{t_i}\})$ is the Conway potential (Alexander polynomial, Reidemeister torsion)

Nonabelian case

- Now let ρ be a representation with nonabelian image.
- $F_r(L, \rho, \{\mu_i\})$ is a deformation of the ADO invariant discussed previously.
- Idea is that the t_i are now the eigenvalues λ_i.
- The novely in the BGPR construction is that we can use nonabelian ρ.
- In the special case $r = 2$ we can say explicitly what we mean by "a deformation."

Torsions of link exteriors

Here's a related abelian/nonabelian link invariant.

- The Reidemeister torsion of $S^3 \setminus L$ is constructed using the ρ-twisted homology $H_*(S^3 \setminus L; \rho)$.
- For ρ sufficiently nontrivial, $H_*(S^3 \setminus L, \rho)$ is acyclic and we can extract a number $\tau(L, \rho)$, the *torsion*.
- For abelian representations $\rho(x) = t$ we get a Laurent polynomial, the Alexander polynomial.
- For abelian representations $\rho(x) = \begin{pmatrix} t & \\ & t^{-1} \end{pmatrix}$ we get the square of the Alexander polynomial.
- For nonabelian representations we get the *twisted* or *nonabelian* Reidemeister torsion.

BGPR invariant versus torsion

Theorem [McP20]

Let L be a link in S^3 and $\rho : \pi_1(S^3 \setminus L)$ a representation such that $\rho(x)$ never has 1 as an eigenvalue for any meridian x of L. Then

$$F_2(L, \rho, \{\mu_i\})F_2(\overline{L}, \overline{\rho}, \{\mu_i\}) = \tau(L, \rho)$$

for any choice of roots μ_i.

Here $\overline{L}$ is the mirror image of L.

$r = 2$ BGPR is a nonabelian Conway potential

One way to understand this theorem:

- If you compute the torsion using the abelian represenation

$$x \mapsto \begin{pmatrix} t & \\ & t^{-1} \end{pmatrix}$$

  it factors into two pieces because the matrix has two blocks. Each piece is equal to the Conway potential of the link.
- For a nonabelian representation, it is not obvious how to factor the torsion into two pieces. But this is exactly what the BGPR invariant does.
- Therefore we could call $F_2(L, \rho)$ a *nonabelian* or *twisted* Conway potential.

Significance

- Torsions are a useful invariant, so this indicates that holonomy invariants should be useful too.
- For example, twisted Alexander polynomials (which are closely related) are quite useful in knot theory.
- It is possible to compute the hyperbolic volume of a knot complement from an asympotic limit of hyperbolically-twisted torsions.
- I am hopeful that a relationship between F_r and the torsions for $r > 2$ can be developed to take advantage of this.

How to construct the BGPR invariant

Quantum $\mathfrak{sl}_2$

Definition

$\mathcal{U}_q = \mathcal{U}_q(\mathfrak{sl}_2)$ is the algebra over $\mathbb{C}(q)$ generated by $K^{\pm 1}, E, F$ with relations

$$KE = q^2 EK, \quad KF = q^{-2} FK, \quad EF - FE = \frac{K - K^{-1}}{q - q^{-1}}$$

- This is a q-analogue of the universal enveloping algebra of $\mathfrak{sl}_2$, with $K = q^H$.
- The center of $\mathcal{U}_q$ is generated by the quantum Casimir element

$$\Omega := (q - q^{-1})^2 FE + qK - q^{-1}K^{-1}$$

Quantum $\mathfrak{sl}_2$ at a root of unity

Set $q = \xi = \exp(\pi i / r)$ a $2r$th root of 1.

Facts

1. $\mathcal{U}_\xi$ is rank r^2 over the central subalgebra
$$\mathcal{Z}_0 := \mathbb{C}[K^r, K^{-r}, E^r, F^r]$$
2. $\mathcal{Z}_0$ is a commutative Hopf algebra, so it's the algebra of functions on a group. Specifically,
$$\operatorname{Spec} \mathcal{Z}_0 \cong \mathrm{SL}_2(\mathbb{C})^*$$
3. The center of $\mathcal{U}_\xi$ is generated by $\mathcal{Z}_0$ and the Casimir Ω (subject to a polynomial relation.)

We will get to the difference between $\mathrm{SL}_2(\mathbb{C})^*$ and $\mathrm{SL}_2(\mathbb{C})$ in a bit.

Grading on representations

- Closed points $\chi \in \operatorname{Spec} \mathcal{Z}_0$ are homomorphisms $\chi : \mathcal{Z}_0 \to \mathbb{C}$.
- We associate

$$\begin{pmatrix} \kappa & -\epsilon \\ \phi & (1-\epsilon\phi)\kappa^{-1} \end{pmatrix} \in \mathrm{SL}_2(\mathbb{C})$$

$$\leftrightarrow$$

$$\chi(K^r) = \kappa, \quad \chi(E^r) = \frac{\epsilon}{(q-q^{-1})^r}, \quad \chi(F^r) = \frac{\phi/\kappa}{(q-q^{-1})^r}$$

- A representation V with $\mathrm{SL}_2(\mathbb{C})$-grading χ is one where every $Z \in \mathcal{Z}_0$ acts by $\chi(Z)$. We say V has *character* χ.

Grading on representations

Theorem

If the the matrix associated to χ does not have ± 1 as an eigenvalue, then:

- Every representation with character χ is projective, irreducible, and r-dimensional.
- There are r isomorphism classes of these, parametrized by the action of Ω.

The idea is that we associate a strand with holonomy corresponding to χ to a representation with character χ. We needed the extra data of the choice $\{\mu_i\}$ of roots to know which of the r irreps to pick.

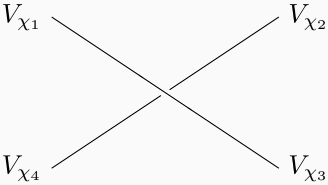
Braiding on representations

- There is an automorphism

$$\check{\mathcal{R}} : \mathcal{U}_\xi \otimes \mathcal{U}_\xi \to \mathcal{U}_\xi \otimes \mathcal{U}_\xi$$

satisfying the braid relations.

- If $\mathcal{U}_\xi$ were quasitriangular $\check{\mathcal{R}}$ would be conjugation by the universal R-matrix followed by swapping the tensor factors, but for technical reasons only the *outer* autormorphism $\check{\mathcal{R}}$ exists.
- $\check{\mathcal{R}}$ acts nontrivially on $\mathcal{Z}_0 \otimes \mathcal{Z}_0$, so it induces a map of modules



corresponding to the colored braid groupoid action on colors. Notice that the isomorphism classes of each strand change.

Problems with the braiding

1. The above action on modules is only defined up to a scalar; we can mostly fix this, but we get the root-of-unity indeterminacy in F_r.
2. The map $(\chi_1, \chi_2) \to (\chi_4, \chi_3)$ is not the conjugation action on $\mathsf{SL}_2(\mathbb{C})$, but something more complicated.

Fixing 2 is harder. It is related to the fact that $\operatorname{Spec} \mathcal{Z}_0$ is really the *Poisson dual group*

$$\mathsf{SL}_2(\mathbb{C})^* := \left\{ \left(\begin{pmatrix} \kappa & 0 \\ \phi & 1 \end{pmatrix}, \begin{pmatrix} 1 & \epsilon \\ 0 & \kappa \end{pmatrix} \right) \right\} \subseteq \mathsf{GL}_2(\mathbb{C}) \times \mathsf{GL}_2(\mathbb{C})$$

$\mathsf{SL}_2(\mathbb{C})^*$ is birationally equivalent as a variety, but not isomorphic as group, to $\mathsf{SL}_2(\mathbb{C})$. The equivalence is

$$(x^+, x^-) \leftrightarrow x^+(x^-)^{-1}$$

Factorized biquandle

- The braid group action

$$(g_1, g_2) \to (g_1^{-1} g_2 g_1, g_1)$$

  on colors is an example of a *quandle*, the conjugation quandle of $\mathrm{SL}_2(\mathbb{C})$.
- A quandle is an algebraic structure that describes colors on arcs of knot diagrams. There are more general ones than conjugation.
- The more complicated action on $\mathrm{SL}_2(\mathbb{C})^*$ colors is a generalization called a *biquandle*.
- It can be shown that the biquandle is a *factorization* of the conjugation quandle of $\mathrm{SL}_2(\mathbb{C})$.

Braid groupoid representations from $\mathcal{U}_\xi$

- Instead of a representation of the colored braid groupoid $\mathbb{B}(\mathrm{SL}_2(\mathbb{C}))$, we get a representation of a different, closely related groupoid $\mathbb{B}(\mathrm{SL}_2(\mathbb{C}))^*$.
- Via the theory of qunandle factorizations developed in [Bla+20], we can use closures of braids in $\mathbb{B}(\mathrm{SL}_2(\mathbb{C}))^*$ to represent $\mathrm{SL}_2(\mathbb{C})$-links.
- Short version: The grading on $\mathcal{U}_\xi$-modules is not quite right, so we have to use a nonstandard coordinate system for representations of link complements.

Modified traces

- To get representations of links (closed braids) we need a way to take traces/closures.
- Problem: The algebra $\mathcal{U}_\xi$ is not semisimple and the *quantum dimensions* of the irreps we want to use are all zero. In particular, all our link invariants will be 0.
- One way to fix this: Take the partial quantum trace of

$$\mathcal{F}(\beta) : V_{g_1} \otimes \cdots V_{g_n} \to V_{g_1} \otimes \cdots \otimes V_{g_n}$$

to get a map $\mathrm{ptr}(\mathcal{F}(\beta)) : V_{g_1} \to V_{g_1}$. (That is, write your link as a *1-1 tangle*.)

Modified traces

- The partial trace $\operatorname{ptr}(\mathcal{F}(\beta)) : V_1 \to V_1$ is an endomorphism of an irreducible module, so by Schur's Lemma there's a scalar with

$$\operatorname{ptr}(\mathcal{F}(\beta)) = \langle \operatorname{ptr}(\mathcal{F}(\beta)) \rangle \operatorname{id}_{V_{g_1}}$$

- The trace of $\operatorname{ptr}(\mathcal{F}(\beta))$ should be $\langle \operatorname{ptr}(\mathcal{F}(\beta)) \rangle$ times the (quantum) dimension of V_{g_1}.
- If we choose *modified dimensions* $\mathrm{d}(V_{g_1})$ correctly, then

$$\langle \operatorname{ptr}(\mathcal{F}(\beta)) \rangle \, \mathrm{d}(V_{g_1})$$

  will be an invariant of the closure L of β.
- There is a theory of *modified traces* due to Geer, Patureau-Mirand, et al. that says how to do this.

Summary

Our algebraic constructions have given us a functor

$$\mathcal{F} : \mathbb{B}(\mathsf{SL}_2(\mathbb{C}))^* \to \mathsf{Rep}(\mathcal{U}_\xi)$$

where $\mathbb{B}(\mathsf{SL}_2(\mathbb{C}))^*$ is a modified version of the groupoid $\mathbb{B}(\mathsf{SL}_2(\mathbb{C}))$ discussed in Part I. To compute the link invariant:

- Write your $\mathsf{SL}_2(\mathbb{C})$-link L as the closure of a braid β in $\mathbb{B}(\mathsf{SL}_2(\mathbb{C}))^*$. (Actually we need to also take some rth roots as well.)
- The modified trace of $\mathcal{F}(\beta)$ is an invariant of L.

Relation to torsions

- Recall that for $r = 2$

$$F_2(L, \rho, \{\mu_i\}) F_2(\overline{L}, \overline{\rho}, \{\mu_i\}) = \tau(L, \rho)$$

  That is, the *norm-square* of F_2 is the torsion.
- To prove this, we work with the squared representation $\mathcal{F} \otimes \overline{\mathcal{F}}$, where $\overline{\mathcal{F}}$ is a mirrored version of $\mathcal{F}$.
- $\overline{\mathcal{F}}$ has inverted gradings, opposite multiplication, and inverted braiding.
- $\mathcal{F} \otimes \overline{\mathcal{F}}$ is a graded version of the quantum double that appears in the correspondence between Reshtikhin-Turaev/Turaev-Viro (surgery/state sum) invaraints.
- The definition of $\mathcal{F} \otimes \overline{\mathcal{F}}$ is more complicated than $\mathcal{F}$, but this representation is in some ways easier to work with.

Twisted burau representations

- The usual torsion can be defined using the *Burau representation* of the braid groupoid $\mathbb{B}$
- The twisted torsion is defined using a *twisted Burau representation* of $\mathbb{B}(\mathrm{SL}_2(\mathbb{C}))$.
- In [McP20] I show that the (super)centralizer of the image of $\mathcal{F} \otimes \overline{\mathcal{F}}$ is naturally isomorphic to the twisted Burau representation.
- Compare Schur-Weyl duality, which computes the tensor decomposition of GL_n representations by showing the centralizers are related to S_n reperesentations.
- Using this result it's not hard to show the desired relationship with the torsion.

Questions? Post them at ncngt.org.

Alternately, I'd love to talk more about this or related mathematics: send me an email and we can get in touch!

These slides are available at esselltwo.com.

References

Christian Blanchet et al. "Holonomy braidings, biquandles and quantum invariants of links with $\mathrm{SL}_2(\mathbb{C})$ flat connections". In: *Selecta Mathematica* 26.2 (Mar. 2020). DOI: 10.1007/s00029-020-0545-0. arXiv: 1806.02787v1 [math.GT].

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