Holonomy invariants of links

Calvin McPhail-Snyder

June 1, 2020

UC Berkeley

Acknowledgements

- I would like to thank Martin Bobb and Allison N. Miller for organizing the Nearly Carbon Neutral Geometric Topology Conference,
- and also to thank Carmen Caprau and Christine Ruey Shan Lee for organizing the session on quantum invariants and inviting me to speak.
- Much of the mathematics I will present is due to Kashaev-Reshetikhin and Blanchet, Geer, Patureau-Mirand, and Reshetikhin, although I will also discuss some of my own work (mostly in the second part.)

Motivation

Quantum holonomy invariants

A *quantum holonomy invariant* is an invariant of topological objects. The adjectives mean:

quantum: it forms part of a topological quantum field theory and/or is constructed using algebraic objects called *quantum groups*

holonomy: instead of just X a topological space it depends on (X, ρ), where $\rho : \pi_1(X) \to G$ is a map into some group G.

Typically we expect it to only depend on the conjugacy class of ρ (*gauge invariance*.)

Terminology

- For geometric applications, G is a Lie group with Lie algebra $\mathfrak{g}$. Then $\rho : \pi_1(X) \to G$ can be described by a flat $\mathfrak{g}$-valued connection whose holonomy is the map ρ.
- Turaev et al. [Tur10] have a notion of *homotopy quantum field theory* for pairs (X, ϕ), where $\phi : X \to Y$ for some fixed Y is considered up to homotopy. For $Y = BG$ a classifying space we recover the map $\rho : \pi_1(X) \to G$.

Motivation I: Better invariants

Why would you want to do this?

- Lots of geometry is captured by a representation into a Lie group.
- For example, if X is a hyperbolic 3-manifold, we have an essentially unique representation $\rho : \pi_1(X) \to \mathsf{SL}_2(\mathbb{C})$.
- By using this extra data, we can get more powerful invariants.
- Compare ordinary Alexander polynomial versus twisted Alexander polynomial: the latter is more powerful. (We will return to this example in part II.)

Motivation II: Volume Conjectures

Why would you want to do this?

Volume Conjecture (Kashaev, Murakami, Murakami, et al.)

- K: hyperbolic knot in S^3 (i.e. a knot whose complement is a hyperbolic 3-manifold of finite volume.)
- $J_n(K)$: nth colored Jones polynomial evaluated at $q = \exp(2\pi i/n)$, normalized so $J_n(\text{unknot}) = 1$.

Then

$$\lim_{n\to\infty} \frac{|\log J_n(K)|}{n} = \frac{\operatorname{Vol}(S^3 \setminus K)}{2\pi}$$

A good overview is [Mur10]. There are many related conjectures and generalizations.

Motivation II: Volume Conjectures

- These conjectures give a relationship between asymptotics of quantum invariants and hyperbolic geometry.
- It is possible to construct holonomy invariants which are deformations of the colored Jones polynomial by the hyperbolic structure.
- The hope is that this relationship can be used to attack the volume conjectures.

Reshetikhin-Turaev invariants

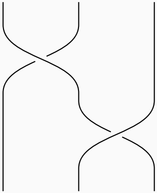
Reminder on the Reshetikhin-Turaev construction

- Before discussing the holonomy version I'll quickly refresh you on the usual RT construction.
- Pick a representation V of a quasitriangular Hopf algebra H (usually H is a quantum group)
- Because it's a Hopf algebra, $V \otimes V$ is also a representation of H
- The quasitriangular structure on H gives a map $c : V \otimes V \to V \otimes V$ called the *braiding*
- c is invertible and satisfies braid relations: If $c_1 = c \otimes \mathrm{id}_V$ and $c_2 = \mathrm{id}_V \otimes c$, then

$$c_1 c_2 c_1 = c_2 c_1 c_2$$

Reminder on the Reshetikhin-Turaev construction

- Assign strands in the braid diagram to V and id_V and crossings to the braiding:



$$\mapsto (\mathrm{id}_V \otimes c^{-1})(c \otimes \mathrm{id}_V)$$

- Get braid group representations $\mathcal{F}_V : \mathbb{B}_n \to \mathrm{GL}(V^{\otimes n})$

Reminder on the Reshetikhin-Turaev construction

- Taking closure of a braid corresponds to a *quantum trace* tr_q generalizing usual trace of linear operators.
- The quantum trace has cyclicity properties like the trace, so it is compatible with Markov moves. Thus:

Theorem

For $L = \langle \beta \rangle$ a braid closure,

$$\mathcal{F}_V(L) = \mathrm{tr}_q\left(\mathcal{F}_V(\beta)\right)$$

is an invariant of L.

Reminder on the Reshetikhin-Turaev construction

Ignoring some issues with framings and orientations:

Jones polynomials

If $H = \mathcal{U}_q(\mathfrak{sl}_2)$ and V is the 2-dimensional irrep, $\mathcal{F}_V(L)$ is the Jones polynomial.

Colored Jones polynomials

If $H = \mathcal{U}_q(\mathfrak{sl}_2)$ and V is the n-dimensional irrep, $\mathcal{F}_V(L)$ is the nth colored Jones polynomial.

HOMFLY-PT polynomials

If $H = \mathcal{U}_q(\mathfrak{sl}_n)$ and V is the n-dimensional irrep, $\mathcal{F}_V(L)$ is the HOMFLY-PT polynomial.

Here $\mathcal{U}_q(\mathfrak{g})$ is a q-analogue of the universal enveloping algebra of $\mathfrak{g}$, a.k.a. a *quantum group*.

Resehtikhin-Turaev holonomy invariants

How to construct holonomy invariants

- I will now sketch how to construct holonomy invariants.
- This is a *Reshetikhin-Turaev* or *surgery* construction. There are also *Turaev-Viro* or *state-sum* approaches.
- I will just describe the process for links in S^3, but there are examples of full 3-2-1 holonomy TQFTs.
- More specifically, I will describe what sort of algebraic machinery you need to get these invariants.
- My second talk will show one way to construct this machinery, due to Blanchet, Geer, Patureau-Mirand, and Reshetikhin [Bla+20]

Representations of knot groups

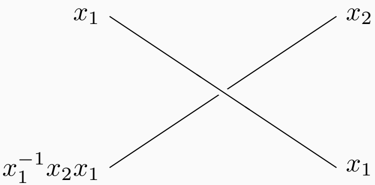
How can we describe the link and its group representation?

- Let $L = \langle \beta \rangle$ be the closure of a braid β on n strands
- The Wirtinger presentation of $\pi_1(S^3 \setminus L)$ has n generators $x_1, \ldots, x_n$ which are meridians for each strand
- Can describe $\rho : \pi_1(S^3 \setminus L) \to G$ by picking $\rho(x_i)$ for each i
- That is, $\rho(x_i)$ is the holonomy around strand i
- How do we know ρ satisfies the relations?

Braid action on free group

The braid group $\mathbb{B}_n$ acts on the free group $\langle x_1, \ldots, x_n \rangle$ by

$$\sigma_i \cdot x_j = \begin{cases} x_i^{-1} x_{i+1} x_i & j = i \\ x_i & j = i+1 \\ x_j & \text{otherwise} \end{cases}$$



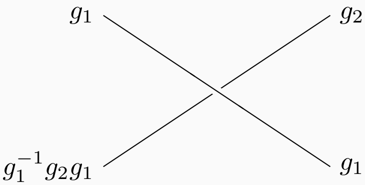
If L is the closure of β, $\pi_1(S^3 \setminus L)$ has presentation

$$\langle x_1, \ldots, x_n | x_1 = \beta \cdot x_1, \ldots, x_n = \beta \cdot x_n \rangle$$

so it is sufficient that ρ satisfies these relations.

Colored braids

Let G be a group. A *G-colored braid* is a braid β on n strands and a tuple $(g_1, \cdots, g_n)$ of elements of G. The braids act on the colors by the rule



(compare the relations for the Wirtinger presentation.) We write this as

$$\sigma_1 : (g_1, g_2) \to (g_1^{-1}g_2g_1, g_1)$$

and more generally $\beta : (g_1, \ldots, g_n) \to (g_1', \ldots, g_n')$.
It makes sense to take the closure of β when it's an endomorphism.

The colored braid groupoid

- G-colored braids form a *groupoid* $\mathbb{B}(G)$ with:

  objects tuples $(g_1, \ldots, g_n)$

  morphisms braids, with the conjugation action on colors:

$$\sigma_1 : (g_1, g_2) \to (g_1^{-1} g_2 g_1, g_1)$$

- A groupoid is like a group, but composing two morphisms is not always defined
- The union $\mathbb{B} = \mathbb{B}_1 \cup \mathbb{B}_2 \ldots$ of the usual braid groups is a groupoid (one object for each number of strands.)

Ordinary braids versus colored braids

	Ordinary braid group(oid) $\mathbb{B}$	G-colored braid groupoid $\mathbb{B}(G)$
Objects	$1, 2, \ldots$	tuples $(g_1, \ldots, g_n)$
Morphisms	braids	braids (acting nontrivially on objects)
Closures	links in S^3	links in S^3 with maps $\pi_1 \to G$

- **Conclusion:** To get holonomy invariants, we want a G-graded version of the RT construction.
- This means a functor $\mathbb{B}(G) \to \mathrm{Rep}(H)$ for some Hopf algebra H with extra structure.

G-graded Reshetikhin-Turaev

- Instead of assigning each strand an H-module V, we need a *family* of modules V_g for $g \in G$
- The braiding is no longer a map $V \otimes V \to V \otimes V$, but a map

$$V_{g_1} \otimes V_{g_2} \to V_{g_1^{-1} g_2 g_1} \otimes V_{g_1}$$

- Instead of a braided monoidal category, we want a G-graded braided monoidal category.

How to get a G-graded category

Here's one way:

- Pick a Hopf algebra H with a big central subalgebra Z_0.
- Z_0 is a commutative Hopf algebra, i.e. the algebra of functions on a group $G = \operatorname{Spec} Z_0$.
- (Closed) points of G are characters $\chi : Z_0 \to \mathbb{C}$.
- A module V_χ with grading $\chi \in G$ is one where $z \in Z_0$ acts by $\chi(z)$.
- For example, if H is finite-rank over Z_0, we can set $V_\chi = H/(\ker \chi)$.
- The example I have in mind is a quantum group $\mathcal{U}_\xi(\mathfrak{g})$ when $q = \xi = \exp(2\pi i/\ell)$ is a root of unity.

More details in part II.

Questions? Post them at ncngt.org.

Alternately, I'd love to talk more about this or related mathematics: send me an email and we can get in touch!

These slides are available at esselltwo.com.

References

Christian Blanchet et al. "Holonomy braidings, biquandles and quantum invariants of links with $SL_2(\mathbb{C})$ flat connections". In: *Selecta Mathematica* 26.2 (Mar. 2020). DOI: 10.1007/s00029-020-0545-0. arXiv: 1806.02787v1 [math.GT].

Hitoshi Murakami. *An Introduction to the Volume Conjecture*. Jan. 31, 2010. arXiv: 1002.0126v1 [math.GT].

Vladimir Turaev. *Homotopy Quantum Field Theory (EMS Tracts in Mathematics)*. European Mathematical Society, 2010. ISBN: 978-3-03719-086-9.