Making the Jones polynomial more geometric

Calvin McPhail-Snyder

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Duke University

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  - B Patureau-Mirand
  - N Reshetikhin
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  - V Turaev
- I will make more specific references later on.

Motivation

- Quantum invariants like the Jones polynomial are defined in an algebraic way.
- However, there is now a lot of interest in what they say about the geometry of knots and manifolds.
- I want to talk about a research program to address these questions by constructing more geometric quantum invariants.

Plan of the talk

- Discuss quantum invariants with examples, so I can explain what I mean by “algebraic”.
- State a conjectured relationship to geometry.
- Give some idea of how to twist the quantum invariants by geometric data.

Quantum invariants

What is a quantum invariant?

- A knot invariant is a function

$$\{\text{knots}\} \to \text{ numbers, polynomials, etc.}$$

- For our purposes, a quantum invariant is a topological invariant constructed using the representation theory of quantum groups.
- Generally quantum invariants appear as part of topological quantum field theories (TQFTs).

Example: the Jones polynomial

Quantum $\mathfrak{sl}_2$

$\mathcal{U}_q = \mathcal{U}_q(\mathfrak{sl}_2)$ is an algebra over $\mathbb{C}(q, q^{-1})$ that we can think of as a q-analogue of the universal enveloping algebra of $\mathfrak{sl}_2$.
For q not a root of unity, it acts a lot like $\mathfrak{sl}_2$.
In particular, there is one[1] representation of dimension $N = 1, 2, \ldots$ which we call V_N.

[1]Well, two, but they are almost identical

Highest-weight representations

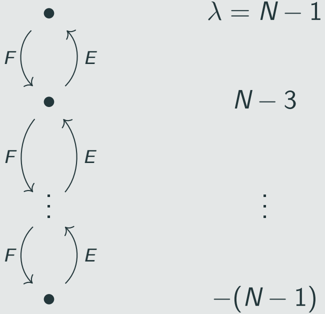
- $\mathfrak{sl}_2$ has generators E, F, H, relations

$$[E, F] = H$$
$$[H, E] = 2E$$
$$[H, F] = -2F$$

- *Weight modules* decompose into H-eigenspaces
- A *highest-weight module* is generated by v with $Ev = 0$, $Hv = \lambda v$ (it has highest weight λ)

Fact

Any finite-dimensional weight module of dimension N looks like



Quantum highest-weight representations

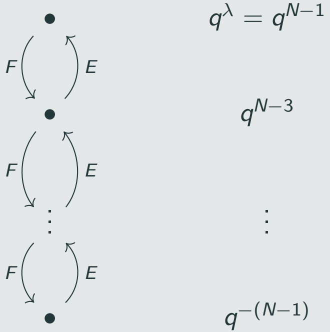
- For $\mathcal{U}_q(\mathfrak{sl}_2)$, replace H with $K = q^H$

$$[E, F] = \frac{K - K^{-1}}{q - q^{-1}}$$
$$KE = q^2 EK$$
$$KF = q^{-2} FK$$

- Weight modules are still described by highest weights
- Important difference: $\mathcal{U}_q$ is *non-cocommutative*: $V \otimes W$ and $W \otimes V$ are different representations.

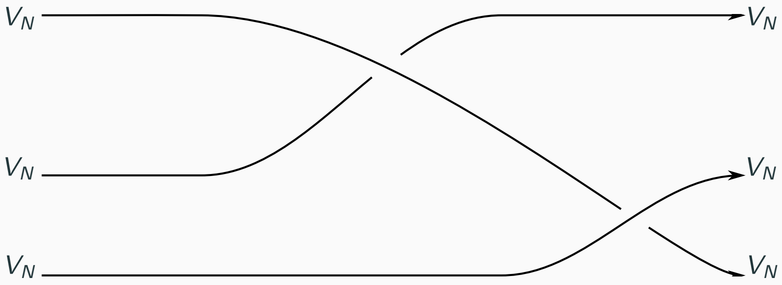
Fact

Any finite-dimensional weight module of dimension N looks like



Braid group representations

Consider a braid β:



Want a $\mathcal{U}$-module map $\mathcal{V}_N(\beta) : V_N^{\otimes 3} \to V_N^{\otimes 3}$.
Key ingredient: value $\mathcal{V}_N(\sigma) : V_N \otimes V_N \to V_N \otimes V_N$ on a braid generator.

Example

The braid above will be mapped to

$$\mathcal{V}_N(\beta) = (\mathcal{V}_N(\sigma)^{-1} \otimes \mathrm{id}_{V_N})(\mathrm{id}_{V_N} \otimes \mathcal{V}_N(\sigma))$$

Getting the braiding

For any $\mathcal{U}_q$-modules V, W, define $\sigma_{V,W} : V \otimes W \to W \otimes V$ by

$$\sigma_{V,W}(x) = \tau(\mathbf{R} \cdot x)$$

where $\mathbf{R} \in \mathcal{U}_q \otimes \mathcal{U}_q$ is the *universal R-matrix*[2] and $\tau(v \otimes w) = w \otimes v$.

Theorem

σ satisfies braid relation

$$(\sigma \otimes \mathrm{id})(\mathrm{id} \otimes \sigma)(\sigma \otimes \mathrm{id}) = (\mathrm{id} \otimes \sigma)(\sigma \otimes \mathrm{id})(\mathrm{id} \otimes \sigma).$$

In terms of $\mathbf{R}$, this is the "Yang-Baxter relation"

Corollary

Any $\mathcal{U}_q$-module V determines a braid group representation.

[2]Actually it's in a sort of completion of $\mathcal{U}_q \otimes \mathcal{U}_q$. This will come up later.

Example: the Jones polynomial

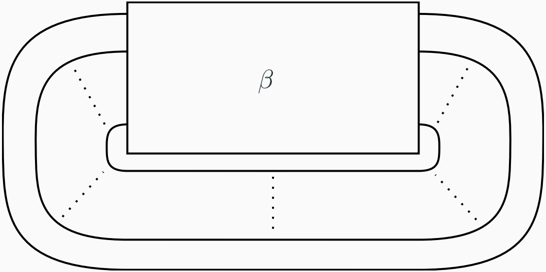
- We want to compute the Jones polynomial of L. Let β be a braid on on b strands whose closure is L.
- $\mathcal{V}_2(\beta)$ is a map $V_2^{\otimes b} \to V_2^{\otimes b}$.
- Determined by value $\mathcal{V}_2(\sigma) : V_2 \otimes V_2 \to V_2 \otimes V_2$ on braid generators.
- $\mathcal{V}_2(\sigma)$ is a 4×4 matrix with entries in $\mathbb{C}[q, q^{-1}]$.
- Explicitly given by action of

$$\mathbf{R} = q^{H \otimes H} \sum_{n=0}^{\infty} c_n E^n \otimes F^n$$

on $V_2 \otimes V_2$.

Taking closures

We think of the closure

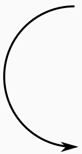


as a trace, and if the closure of β is L, then (up to some framing issues)

$$\mathrm{tr}_q \, \mathcal{V}_2(\beta) = \mathrm{V}_2(\beta)$$

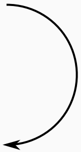
is the Jones polynomial of L.

Cups and caps



$$\mathrm{coev}_V : \mathbb{k} \to V^* \otimes V$$

$$\mathrm{coev}_V(1) = \sum_i v^i \otimes v_i$$



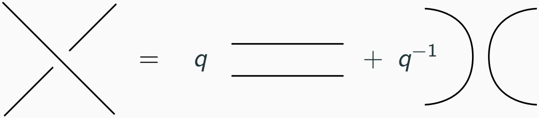
$$\mathrm{ev}_V : V \otimes V^* \to \mathbb{k}$$

$$\mathrm{ev}_V(v \otimes f) = f(v)$$

Notice these don't match. Other orientations are a bit more complicated, leads to a twist in definition of quantum trace tr_q.

Graphical calculus

Can write action of **R** in terms of other maps:



which can be used to define the Jones polynomial without using quantum groups at all.

An example

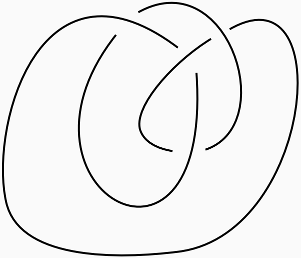


Figure eight knot 4_1

$$\mathrm{V}_2(4_1) = q^{-4} - q^{-2} + 1 - q^2 + q^4$$

Computing the Jones polynomial

To compute the Jones polynomial $\mathrm{V}_2(L)$ of a link L :

- Represent L as the closure of a braid β on b strands
- Compute the $2^b \times 2^b$ matrix $\mathcal{V}_2(\beta)$
- Its (quantum) trace is a Laurent polynomial $\mathrm{V}_2(L)$ in q^2
- This is an invariant[3] of L called the Jones polynomial

This is an example of the *Reshetikhin-Turaev construction* applied to the representation V_2.

[3] Modulo some technicalities about framings that are not important here.

The colored Jones polynomial

- We can repeat the Resethikin-Turaev construction defining $\mathrm{V}_2(L)$ with any representation of $\mathcal{U}_q$ (or any quantum group).

Definition

The quantum invariant assigned to a link L by the N-dimensional $\mathcal{U}_q$-module V_N is the *Nth colored Jones polynomial* $\mathrm{V}_N(L)$.

- We can do this diagrammatically in terms of cables of links, or by using *Jones-Wenzl projectors*

Algebra → topology

This process was very algebraic. I used words like:

- quantum group (a q-analog of a Lie algebra/group)
- trace
- representation (of a group/algebra)

I did *not* use more topological/geometric ideas like

- homology/fundamental groups
- essential surfaces
- geometrization

However, all this algebra still knows about geometry!

Value at roots of unity

We are most interested in particular values for knots K.
Set $\xi = \exp(\pi i/N)$ and normalize so that $\mathrm{V}_N(\text{unknot}) = 1$.

Definition

The complex number

$$\mathrm{J}_N(K) = \mathrm{V}_N(K)|_{q=\xi}$$

is called the *Nth quantum dilogarithm* of K.

Why the name? We will explain later.

Value at roots of unity

Figure-eight knot

Set $\{k\} = \xi^k - \xi^{-k}$. Then

$$\mathrm{J}_N(4_1) = \sum_{j=0}^{N-1} \prod_{k=1}^{j} \{N-k\}\{N+k\}.$$

- Computing these closed formulas for all N is hard!
- One reason: if K is presented as the closure of a braid on b strands, then computing $\mathrm{J}_N(K)$ involves the trace of a $N^b \times N^b$ matrix.
- This one comes from writing 4_1 as surgery on the Borromean rings.

So far, only algebra

- The quantum dilogarithm (and things like it) are algebraic: coming from representation theory.
- What does it *mean* that $\mathrm{J}_N(4_1) = \sum_{j=0}^{N-1} \prod_{k=1}^{j} \{N-k\}\{N+k\}$?

Geometric connections

Theorem

$$2\pi \lim_{N\to\infty} \frac{\log |\mathrm{J}_N(4_1)|}{N} = 2.02988\ldots = \mathrm{Vol}(4_1)$$

where $\mathrm{Vol}(K)$ *is the volume of the complete hyperbolic structure of* $S^3 \setminus K$.

Conjecture (Volume conjecture [Kas97; MM01])

For any hyperbolic knot K*,*

$$2\pi \lim_{N\to\infty} \frac{\log |\mathrm{J}_N(K)|}{N} = \mathrm{Vol}(K).$$

- There are versions for complex volume, for knots in 3-manifolds, for 3-manifolds...
- In every case where the left-hand limit is known to exist the conjecture holds.

Hyperbolic geometry and knot complements

Definition

A knot K in S^3 is hyperbolic if it admits a complete finite-volume constant-negative curvature metric on its complement. Can be described as a faithful discrete representation

$$\rho : \pi_1(S^3 \setminus K) \to \mathrm{PSL}_2(\mathbb{C}) = \mathrm{Isom}(\mathbb{H}^3).$$

Up to conjugacy ρ is a homotopy invariant of $S^3 \setminus K$!

Theorem

Every knot in S^3 is:

- *a torus knot,*
- *a satellite knot (composed of other knots),*
- *hyperbolic.*

Hence most "irreducible" knots are hyperbolic.

How does J_N know about hyperbolic geometry?

How does J_N know about hyperbolic geometry?

- It's a conjecture, so no one really knows.
- I can now get to the main point of my talk: a program aimed at answering this sort of question.
- Along the way I hope we can define some new, even better knot invariants.

Holonomy invariants

The idea

- To describe geometry of a topological space X, pick a (conjugacy class of) representations $\pi_1(X) \to G$ for G a Lie group
- For example, a hyperbolic structure on a 3-manifold X is given by a

$$\rho : \pi_1(X) \to \mathrm{Isom}(\mathbb{H}^3) = \mathrm{PSL}_2(\mathbb{C})$$

  usually called the *holonomy representation*.
- We focus on $X = S^3 \setminus K$ a knot complement and $G = \mathrm{SL}_2(\mathbb{C})$.
- Sometimes (especially in physics contexts) we view this data as a flat $\mathfrak{sl}_2$-connection on X.

The idea

Definition

A $\mathrm{SL}_2(\mathbb{C})$-*holonomy invariant* of knots gives a scalar $F(K,\rho) \in \mathbb{C}$, where $\rho : \pi_1(S^3 \setminus K) \to \mathrm{SL}_2(\mathbb{C})$. It should depend only on the conjugacy class (gauge class) of ρ.

From now on, we say *holonomy invariant* and assume $G = \mathrm{SL}_2(\mathbb{C})$.

Another perspective

A holonomy invariant assigns a function $F(K,-) : \mathfrak{X}(K) \to \mathbb{C}$ to every knot, where $\mathfrak{X}(K)$ is the $\mathrm{SL}_2(\mathbb{C})$-character variety of K.

Examples of holonomy invariants

Torsion

The Reidemeister torsion $\tau(K, \rho) = \tau(S^3 \setminus K, \rho)$ depends on K and $\rho \in \mathfrak{R}(K)$. It is gauge-invariant, so we get a function

$$\tau(K, -) : \mathfrak{X}(K) \to \mathbb{C}$$

i.e. a holonomy invariant.

Examples of holonomy invariants

Complex volume

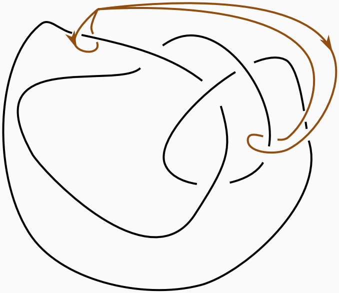
The *complex volume* of a hyperbolic knot

$$\mathrm{Vol}(K) + i\,\mathrm{CS}(K) \in \mathbb{C}/i\pi^2\mathbb{Z}$$

can be computed by evaluating a certain characteristic class of flat $\mathrm{PSL}_2(\mathbb{C})$-bundles on the finite-volume hyperbolic structure of $S^3 \setminus K$. We can think of this as a holonomy invariant by evaluating that class on other elements of $\mathfrak{X}(K)$.

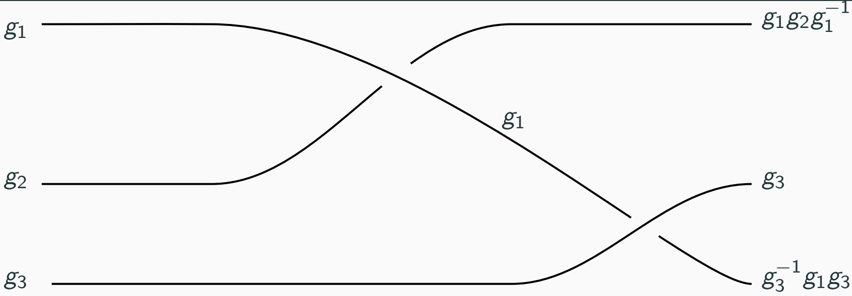
The knot group

- K a knot in S^3
- $\pi_K = \pi_1(S^3 \setminus K)$ is finitely generated, say by *meridians*
- All meridians of K are conjugate
- For a link L there's one conjugacy class for each component



Two meridians of the figure-eight knot

Colored braids



- Label $g \in \mathrm{SL}_2(\mathbb{C})$ gives holonomy of meridian around it
- Braid group acts nontrivially on g_i , get a *groupoid* with

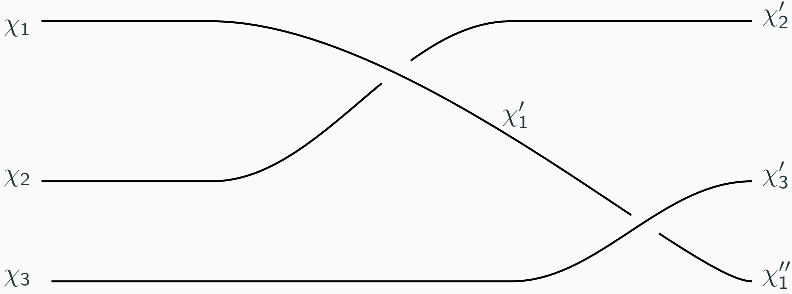
  objects tuples $(g_1, \ldots, g_n)$ of group elements $g_i \in \mathrm{SL}_2(\mathbb{C})$

  morphisms braids $\beta : (g_1, \ldots, g_n) \to (g_1', \ldots, g_n')$

  Closures of endomorphisms give links L with $\rho : \pi_L \to \mathrm{SL}_2(\mathbb{C})$
- To get invariants, want a representation (functor) from the colored braid groupoid

Well, not quite

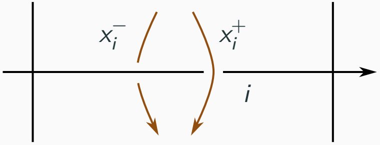
We need to use a more complicated description:



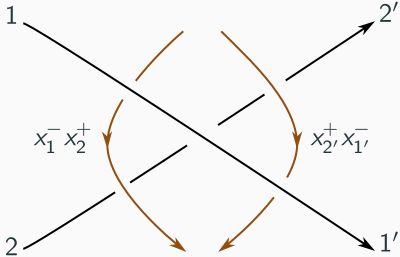
Notice *both* labels change at a crossing. What does this mean?

- Usual description (Wirtinger presentation) of knot group from a diagram has one generator for each arc.
- We instead want a *groupoid* with two generators for each segment.
- Path above a segment labeled by χ gives χ^+, path below gives χ^-

Fundamental groupoid



The generators associated to segment i



There are relations at each crossing, such as the above

- Seems more complicated: isn't in practice.
- More natural in relation to hyperbolic geometry.

Factorized groups

- Labels are

$$\begin{aligned}\chi &= (\chi^+, \chi^-) \\ &= \left(\begin{bmatrix} \kappa & 0 \\ \phi & 1 \end{bmatrix}, \begin{bmatrix} 1 & \epsilon \\ 0 & \kappa \end{bmatrix} \right) \in \mathsf{SL}_2(\mathbb{C})^* \subset \mathsf{GL}_2(\mathbb{C}) \times \mathsf{GL}_2(\mathbb{C})\end{aligned}$$

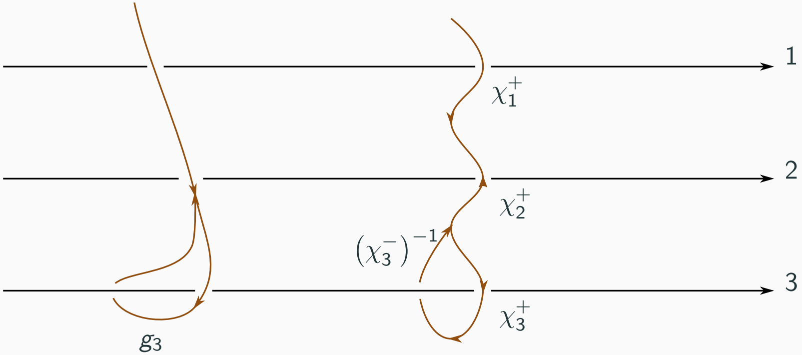
- Get a birational map $\psi : \mathsf{SL}_2(\mathbb{C})^* \to \mathsf{SL}_2(\mathbb{C})$ by

$$\psi(\chi) = \chi^+(\chi^-)^{-1} = \begin{bmatrix} \kappa & -\epsilon \\ \phi & \kappa^{-1}(1 - \epsilon\phi) \end{bmatrix}$$

Think of $\psi(\chi)$ as the holonomy around the strand labeled by χ

Factorized groups

More generally:



$$g_3 = \chi_1^+ \chi_2^+ \chi_3^+ (\chi_3^-)^{-1} (\chi_2^+)^{-1} (\chi_1^+)^{-1}$$

Factorized groups

The pictures on the last slide extend to a functor

$$\Psi : \mathbb{B}(\mathsf{SL}_2(\mathbb{C})^*) \to \mathbb{B}(\mathsf{SL}_2(\mathbb{C}))$$

from the category of $\mathsf{SL}_2(\mathbb{C})^*$-colored braids to the category of $\mathsf{SL}_2(\mathbb{C})$-colored braids. Ψ is not surjective, but:

Theorem (Blanchet, Geer, Patureau-Mirand, and Reshetikhin [Bla+20])

Every $\mathsf{SL}_2(\mathbb{C})$*-colored braid is conjugate to one in the image of* Ψ *(i.e. that can be represented in terms of the* χ*.)*

In their language, we have a *generic biquandle factorization* of the *conjugation quandle* of $\mathsf{SL}_2(\mathbb{C})$.

What now?

To construct a holonomy version of $\mathrm{J}_N(K)$ (the Nth colored Jones polynomial at a root of unity) we want:

1. A family V_χ of N-dimensional modules parametrized by $\mathrm{SL}_2(\mathbb{C})^*$
2. A braiding

$$c : V_{\chi_1} \otimes V_{\chi_2} \to V_{\chi_{2'}} \otimes V_{\chi_{1'}}$$

respecting the transformation rules for the χ_i.

Recovering $\mathrm{J}_N(K)$

We want to be able to recover the ordinary invariant. One way is to ask that $V_{\pm \mathrm{id}} = V_N$ is the original N-dimensional highest weight module at $q = \xi$.

Constructing the holonomy invariant

The center for generic q

Recall that simple A-modules are parametrized by the center of A.

Generic q

For q not a root of unity, the center of $\mathcal{U}_q$ is generated by the Casimir

$$\Omega = \frac{qK + q^{-1}K^{-1}}{(q - q^{-1})^2} + FE$$

whose action determines the isomorphism class of any finite-dimensional simple $\mathcal{U}_q$-module.

The center at a root of unity

Key algebra fact

When $q = \xi = \exp(\pi i/N)$ $\mathcal{U}_\xi$ has a large center.

The center at a root of unity

- At $q = \xi$, get central subalgebra $\mathcal{Z}_0 = \mathbb{C}[E^N, F^N, K^{\pm N}]$
- For[4] central characters $\chi : \mathcal{Z}_0 \to \mathbb{C}$,

$$\chi \in \operatorname{Spec} \mathcal{Z}_0 \leftrightarrow \left(\begin{bmatrix} \chi(K^N) & 0 \\ \chi(K^N F^N) & 1 \end{bmatrix}, \begin{bmatrix} 1 & \chi(E^N) \\ 0 & \chi(K^N) \end{bmatrix} \right)$$

$$\leftrightarrow \psi(\chi) = \begin{bmatrix} \chi(K^N) & -\chi(E^N) \\ \chi(K^N F^N) & \chi(K^N) - \chi(K^N E^N F^N) \end{bmatrix} \in \mathsf{SL}_2(\mathbb{C})$$

- Full center is $\mathcal{Z} = \mathcal{Z}_0[\Omega]/(\text{polynomial relation})$
- Action of central Casimir Ω given by Nth root of an eigenvalue of $\psi(\chi)$
- Characters $\chi : \mathcal{Z} \to \mathbb{C}$ are in bijection with simple $\mathcal{U}_\xi$-modules.

[4] There are some normalizations I'm suppressing. Can also remove them by using a different presentation of $\mathcal{U}_q$.

Theorem

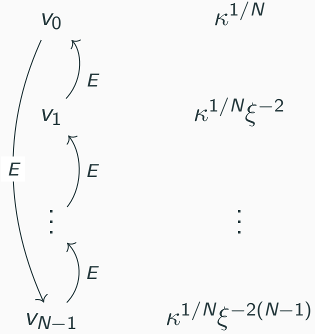
For any $\chi : \mathcal{Z}_0 \to \mathbb{C}$ there are N simple projective $\mathcal{U}_\xi$-modules $V_{\chi,\mu}$ with central character χ_0.

Isomorphism class is determined by *fractional eigenvalue* μ with

$$\mu^N + \mu^{-N} = \operatorname{tr} \psi(\chi)$$

i.e. by an Nth root of an eigenvalue of the holonomy around a meridian colored by $\chi \in \operatorname{Spec} \mathcal{Z}_0 = \mathrm{SL}_2(\mathbb{C})^*$.

$$\chi(K^N) = \kappa, \chi(E^N) = \epsilon$$



$$E \cdot v_k = v_{k-1}$$

$$E \cdot v_0 = \epsilon v_{N-1}$$

A holonomy invariant from the $V_{\chi,\mu}$

Theorem (Blanchet, Geer, Patureau-Mirand, and Reshetikhin [Bla+20])

There is a holonomy invariant $\mathrm{BGPR}_N\, L, \rho$ *assigning* $V_{\chi,\mu}$ *to strands, well-defined up to a* N^2*th root of unity.*

Because of Casimirs before, also depends on a fractional eigenvalue μ of holonomy around each component of L. Natural to consider ρ in *extended character variety*

$$\mathfrak{X}_N(L) \to \mathfrak{X}(L)$$

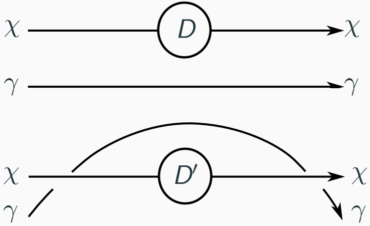
an $N^{|\text{components}(L)|}$-fold cover of the usual character variety.

Issues defining BGPR

- Not every $\rho : \pi_L \to \mathsf{SL}_2(\mathbb{C})$ can be written using $\mathsf{SL}_2(\mathbb{C})^*$-coordinates, but can show it's generically true. (Gave this theorem earlier.)
- Quantum dimensions of $V_{\chi,\mu}$ vanish, so normal way of taking closures gives 0 for every link. Fixable with *modified dimensions* of Geer, Patureau-Mirand, and Turaev [GPT09].
- The braiding is hard to define.

Gauge invariance

However, gauge invariance is easy to prove. Two pictures explain why:



Here D' is gauge-equivalent to D.

The braiding for cyclic modules

- Recall braiding was given by action of

$$\mathbf{R} = q^{H \otimes H} \sum_{n=0}^{\infty} c_n E^n \otimes F^n$$

- Since $E \otimes F$ doesn't act nilpotently on $V_{\chi_1} \otimes V_{\chi_2}$, this doesn't converge!
- Can fix by understanding automorphism

$$\mathcal{R} : \mathcal{U}_\xi^{\otimes 2} \to \mathcal{U}_\xi^{\otimes 2}$$

  given by conjugation by $\mathbf{R}$.
- Still doesn't give an explicit formula for braiding.

Improving BGPR

Theorem (My PhD thesis [McP21a])

There is a version $\mathrm{J}_N(L, \rho)$ of BGPR defined up to a $2N$th root of unity, including an explicit formula for the braiding matrices.

- Currently working on how to define the phase absolutely; may require some extra structure.
- Coordinates used to compute braiding have direct connection to hyperbolic geometry via *octahedral decomposition* of the link complement
- Braiding factors into four *cyclic quantum dilogarithms*
- $\mathrm{J}_N(L, \rho)$ should be part of Chern-Simons TQFT with noncompact gauge group $\mathrm{SL}_2(\mathbb{C})$; usual case is compact group $\mathrm{SU}(2)$

Application to the volume conjecture

Conjecture

1. *Asymptotics of* $\mathrm{J}_N(K, \rho_{hyp})$ *determine* $\mathrm{Vol}(K)$
2. *Can relate asymptotics of colored Jones* $\mathrm{J}_N(K, (-1)^{N+1})$ *and hyperbolically-twisted colored Jones* $\mathrm{J}_N(K, \rho_{hyp})$

Together, would give the volume conjecture.

$\mathrm{J}_N(K, (-1)^{N+1})$	$\leftrightarrow$	$\mathrm{J}_N(K, \rho_{\mathrm{hyp}})$
invariant in volume conjecture		should know about $\mathrm{Vol}(K)$

Examples of $\mathrm{V}_N(K, \rho)$

- Unfortunately I don't have many examples.
- Issue with braiding normalization made BGPR very hard to compute
- Definition of J_N is recent and not quite done.
- Can say things in some special cases.

The abelian case

Kashaev's quantum dilogarithm

When $\rho = (-1)^{N+1}$ is $\pm$ the trivial representation,

$$\mathrm{J}_N(K, (-1)^{N+1}) = \mathrm{J}_N(K)$$

is the quantum dilogarithm, i.e. the Nth colored Jones polynomial evaluated at $\exp(2\pi i/N)$.

The Akutsu-Deguchi-Ohtsuki invariant

When $\rho = \alpha_t$ sends every meridian to $\operatorname{diag}(t, t^{-1})$,

$$\mathrm{J}_N(K, \alpha_t) = \mathrm{ADO}_N(t)$$

is the Nth ADO invariant.

The ADO invariant is a higher-order Alexander polynomial. When $N = 2$, it is exactly the Conway potential/Alexander polynomial/abelian Reidemeister torsion.

Relation with the torsion

Theorem (Me [McP21b])

For any link L and $\rho \in \mathfrak{X}_2(L)$ that does not have 1 as an eigenvalue,

$$\mathrm{J}_2(L, \rho)\mathrm{J}_2(\overline{L}, \rho) = \tau(S^3 \setminus L, \rho)$$

where $\overline{L}$ is the mirror image and τ is the Redemeister torsion twisted by ρ.

Proof idea.

There is a Schur-Weyl duality between the braiding for $\mathcal{U}_\xi$ defining J_2 and the twisted Burau representation defining τ. Need to use a "quantum double" to get the norm-square on the left hand side. □

Another holonomy invariant

Quantum hyperbolic invariants

Baseilhac and Benedetti [BB04] constructed *quantum hyperbolic invariants* of 3-manifolds with links inside them via state-sums and triangulations.

- They used quantum dilogarithms, just like in our construction
- Their invariants appear to be closely related to our nonabelian quantum dilogarithm.
- Our version is much more directly related to the Jones polynomial
- Our version gives relation with torions

Another holonomy invariant, with examples

Extending Kashaev and Reshetikhin [KR05], myself, Chen, Morrison, and Snyder [Che+21] constructed a holonomy invariant. Set $\zeta = \exp(2\pi i/\ell)$ for ℓ odd.

Fact

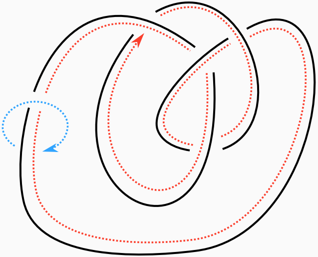
$\mathcal{U}_\zeta / \ker \chi$ is a simple bimodule of dimension N^2 for any $\mathcal{Z}$-character χ.

Theorem

By assigning a strand of a knot diagram with holonomy χ the module $\mathcal{U}_\zeta / \ker \chi$, we get a holonomy invariant $\mathrm{KR}(K, \rho)$ of knots. $\mathrm{KR}(K, -)$ is a rational function on a N-fold cover $\mathfrak{X}_N(K)$ of $\mathfrak{X}(K)$.

For technical reasons it is much easier to define the braiding.

KR for the figure-eight knot



$K = 4_1$

longitude meridian

$$\mathfrak{X}(4_1) = \mathbb{C}[M^{\pm 1}, L^{\pm 1}] / \left\langle (L-1)(L^2 M^4 + L(-M^8 + M^6 + 2M^4 + M^2 - 1) + M^4) \right\rangle$$

$M^{\pm 1}$ are the eigenvalues of the meridian and $L^{\pm 1}$ are the eigenvalues of the longitude.

To get $\mathfrak{X}_N(4_1)$, replace M with $\mu^N = M$

KR for the figure-eight knot

$(L-1)$ factor is the *commutative* component and the other is *geometric*. We compute that, for $N=3$,

$$\mathrm{KR}(K)_{\mathsf{comm}} = \left(\mu^4 + 3\mu^2 + 5 + 3\mu^{-2} + \mu^{-4}\right)^2$$
$$\mathrm{KR}(K)_{\mathsf{geom}} = 3(\mu^2 + \mu^{-2})(\mu + 1 + \mu^{-1})^3(\mu - 1 + \mu^{-1})^3$$

Complete hyperbolic structure of 4_1 complement corresponds to points $\mu = 1, \exp(2\pi i/3), \exp(4\pi i/3)$ on geometric component.

Observation

$\mathrm{KR}(K)_{\mathsf{geom}}$ vanishes for μ a primitive root of unity. Seems to extend to other knots and odd N for $\zeta = \exp(2\pi i/N)$; does not occur for $\xi = \exp(\pi i/N)$ and N even.

Future examples?

It should be possible to repeat this computation with J_N instead of KR and get rational functions on the character variety (or A-polynomial curve).

Questions?

Bonus: Why is it called a quantum dilogarithm?

The dilogarithm

- The dilogarithm is

$$L_2(x) = -\int_0^x \frac{\log(1-z)}{z}\,dz$$

  and Rogers' dilogarithm is

$$L(x) = L_2(x) + \log(1-x)\log(x)/2.$$

  $L(x)$ can be used to compute complex volumes of tetrahedra, hence of manifolds.

- It satisfies the 5-term relation

$$L(x) + L(y) - L(xy) = L\left(\frac{x-xy}{1-xy}\right) + L\left(\frac{y-xy}{1-xy}\right)$$

  which is related to the 3-2 move on triangulations

The quantum dilogarithm

- Faddeev and Kashaev showed the q-series

$$\Psi(x) = \prod_{n=1}^{\infty} (1 - xq^n)$$

  is a q-analog of $L(x)$ and satisfies a noncommutative 5-term relation.

- The *cyclic quantum dilogarithm*

$$L(B, A|n) = \prod_{k=1}^{n} (1 - \xi^{2k} B)/A$$

  for $A^N + B^N = 1$ is a root-of-unity analogue of $\Psi(x)$.

Link invariants from the quantum dilogarithm

- By taking a certain singular limit Kashaev defined his quantum dilogarithm invariant.
- By replacing Rogers dilogarithms $L(x)$ with cyclic dilogarithms $L(B, A|n)$, Baseilhac and Benedetti defined holonomy invariants B_N for triangulated 3-manifolds with links inside them.
- B_N is constructed as a state-sum, with one function $L(B, A|n)$ for each tetrahedron.

The nonabelian quantum dilogarithm

- Even though the definition of J_N appears quite different from B_N, recent computations of the braiding show they are closely related.
- In particular, the braiding defined by $\mathcal{J}_N$ factors into a product of four linear maps, each of which is associated to a tetrahedron in the octahedral decomposition of the knot complement.
- To emphasize the connection with Kashaev's construction and the incorporation of *nonabelian* $\rho \in \mathfrak{X}_N(K)$, we used the name *nonabelian quantum dilogarithm*.

The nonabelian quantum dilogarithm and the torsion

An explicit relationship

Theorem (C. [McP21b])

For any $\rho \in \mathfrak{X}_2(K)$,

$$\mathrm{J}_2(K, \rho)\mathrm{J}_2(\overline{K}, \rho) = \tau(K, \rho)$$

where $\overline{K}$ is the mirror image of K.

Comparing

$$\nabla_K(t)\nabla_{\overline{K}}(t) = \tau(K, \alpha_t)$$

we think of $\mathrm{J}_2(K, \rho)$ as a *nonabelian Conway potential*.

How do we compute the right-hand side? Use the Burau representation.

The Burau representation

Consider colored braids on b strands. Write $\rho = (\chi_1, \ldots, \chi_b)$ for an object of $\mathbb{B}_2(\mathsf{SL}_2(\mathbb{C}))$, equivalently a representation

$$\rho : \pi_1(D_b) \to \mathsf{SL}_2(\mathbb{C})$$

where D_b is a b-punctured disc. Let β be a braid on b stands, i.e. an element of $\mathsf{Map}(D_b, \partial D_b)$. As a colored braid, it becomes a morphism $\beta : \rho \to \rho'$.

Definition

The *Burau representation* is the action on twisted locally-finite homology:

$$\mathcal{B}(\beta) : \mathrm{H}_1(D_b; \rho) \to \mathrm{H}_1(D_b; \rho')$$

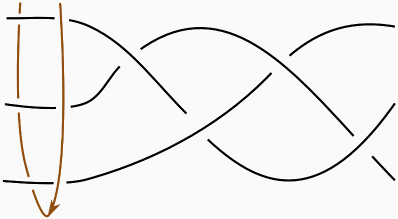
induced by the action of β on D_b.

Computing the torsion

Proposition

If (K, ρ) is the closure of β, then

$$\tau(K, \rho) = \frac{\det(1 - \mathcal{B}(\beta))}{\det(1 - \rho(y))}$$



y is a path around every strand, as above.

Determinant to trace

To make this a trace, let $\bigwedge \mathcal{B}$ be the action on the exterior algebra of homology. Then

$$\mathrm{str}\left(\bigwedge \mathcal{B}(\beta)\right) = \det(1 - \mathcal{B}(\beta)).$$

Here str is the $\mathbb{Z}/2$-graded trace: multiply action on degree k part by $(-1)^k$.

Multiplicity spaces

We want to understand $\mathcal{J}_2(\beta) : \mathcal{J}_2(\rho) \to \mathcal{J}_2(\rho)$, $\rho = (\chi_1, \ldots, \chi_b)$. First we need to understand $\mathcal{J}_2(\rho)$. Use:

Proposition

$$\mathcal{J}_2(\rho) = \bigotimes_{i=1}^{b} V_{\chi_i} \cong X^+ \otimes_{\mathbb{C}} V_{\chi_+} \oplus X^- \otimes_{\mathbb{C}} V_{\chi_-}$$

Here:

- $\chi_\pm$ are characters corresponding to the total holonomy $\rho(y)$
- there are two because there are two choices $\pm\mu$ of fractional eigenvalue for $\rho(y)$
- Action of $\mathcal{J}_2(\beta)$ factors through *multiplicity spaces* $X^\pm$

Schur-Weyl duality

Theorem (Me [McP21b])

There is a subalgebra $\mathfrak{C}_b$ of $\mathcal{U}_\xi^{\otimes b}$ that

1. *(super)commutes with the image of $\Delta\mathcal{U}$ in the tensor power,*
2. *is isomorphic as a vector space to $\bigwedge \mathcal{B}(\chi_1, \dots, \chi_b)$,*
3. *such that the braid group action on $\mathfrak{C}_b \subseteq \mathcal{U}_\xi^{\otimes b}$ agrees with $\mathcal{B}$.*

Compare Schur-Weyl duality between tensor powers of SL_n and the symmetric group.

Computing J_2

Corollary (Wrong)

The $\mathbb{Z}/2$-graded multiplicity space $X = X^+ \oplus X^-$ is isomorphic to $\bigwedge \mathcal{B}(\rho)$. This is compatible with the braid action, so $\mathcal{J}_2(\beta)$ acts on X by $\bigwedge \mathcal{B}(\beta)$.

The theorem about $\tau(K, \rho)$ would follow immediately, except that this is false!

Fixing the idea

- The problem is that $\mathfrak{C}_b$ does not act faithfully on $\mathcal{J}_2(\chi_1, \ldots, \chi_n)$.
- Among other reasons, dimensions don't match.
- To fix, consider a "quantum double"

$$\mathcal{T}_2 = \mathcal{J}_2 \boxtimes \overline{\mathcal{J}}_2$$

- Then the theorem works and

$$\begin{aligned} \tau(K, \rho) &= \mathrm{T}_2(K, \rho) && \text{(by Schur-Weyl)} \\ &= \mathrm{J}_2(K, \rho)\mathrm{J}_2(\overline{K}, \rho) && \text{(by definition)} \end{aligned}$$

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