

When do octahedral decompositions detect a hyperbolic structure?

Calvin McPhail-Snyder
AMS Fall Central Sectional Meeting
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Thank you for the talk invitation!

Most of this talk is about mathematics other people have done.

I want to share it with you, because I found it very helpful when studying geometric (quantum) invariants.

The new theorem (due to me) is mostly about a different perspective on their work.

PLAN OF THE TALK

1. Statement of the main result
2. Octahedral decompositions and coordinates
3. Coordinates from Wirtinger representations
4. More on the main result

BACKGROUND

- Diagram D of a knot K
 - Most of this also works for links and tangles
- **Octahedral decomposition** has one octahedron at each crossing
- Subdivide into an ideal triangulation of $S^3 \setminus (K \cup \{p, q\})$:
 - 4 tetrahedra per crossing: $\mathcal{O}_D^4$
 - 5 tetrahedra per crossing: $\mathcal{O}_D^5$
- Due to Weeks, Kashaev, D. Thurston, W. Thurston (?), et al (?)
- A good reference is [Kim, Kim, and Yoon, 2018]
- $\pi = \pi_1(S^3 \setminus K)$ knot group
- $\rho : \pi \rightarrow \mathrm{SL}_2(\mathbb{C})$ representation (hyperbolic structure)
- To realize ρ , place ideal vertices in $\partial\mathbb{H}^3 = \mathbb{CP}^1$.
- What are the coordinates in terms of the Wirtinger matrices?

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Theorem 1 [Cho, 2018] (boundary parabolic)
[Yoon, 2021] (general)

If we choose a **decoration** and **shadow coloring** then there are explicit formulas for the octahedral coordinates in terms of ρ .

MAIN RESULT

Definition ρ is **detected** by a subdivision if its holonomy is conjugate to ρ and every tetrahedron is nondegenerate

- this means cross-ratio $z_i \notin \{0, 1, \infty\}$
- $z_i \in \mathbb{R}$ is OK
- Positively oriented/geometric tetrahedra not required

Theorem 2 [Cho, 2016]; [Yoon, 2021]; [McPhail-Snyder, 2025], [last is me]

- $\mathcal{O}_D^5$ detects every representation whose meridians avoid $\pm I$.
- $\mathcal{O}_D^4$ detects a decorated ρ exactly when the distinguished fixed points at each crossing are distinct. (Last interpretation is my contribution.)

Theorem 3 [McPhail-Snyder, 2025] [Me] For a hyperbolic knot, $\mathcal{O}_D^4$ detects the complete holonomy exactly when D is **arc-faithful**: adjacent Wirtinger generators x_1, x_2 at each crossing are distinct in π .

WHY SHOULD YOU CARE?

Octahedral coordinates translate between **diagrams** and **triangulations**.

In quantum topology, diagrams are often easier to work with.

I was studying a **quantum hyperbolic invariant** $\mathcal{Z}_N^\psi$ [McPhail-Snyder, 2025] that combines features of the Jones polynomial and hyperbolic volume. To understand it I needed to understand octahedral decompositions.

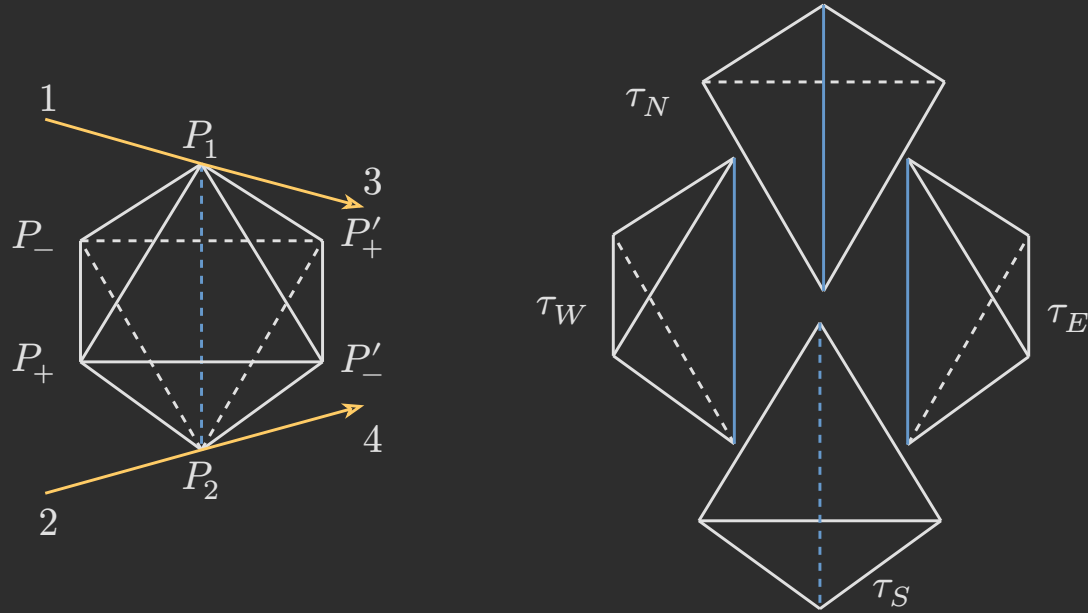
They also lead to a way to define the $\mathrm{SL}_2(\mathbb{C})$ Chern-Simons invariant for tangles.

A diagram determines a **potential function** V_D .

Its (generalized) critical points are solutions of the gluing equations of $\mathcal{O}_D^4$, ie representations detected by $\mathcal{O}_D^4$.

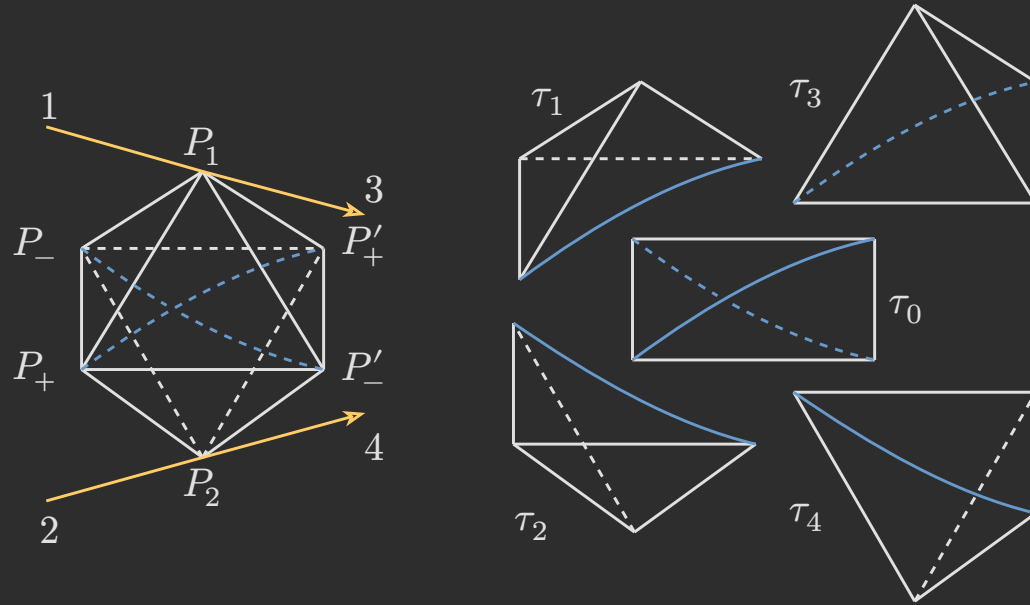
For saddle-point approaches to the **Volume Conjecture** [Yokota, 2000] one wants a nonsingular critical point realizing the complete hyperbolic structure ρ_{hyp} .

FOUR-TERM OCTAHEDRON



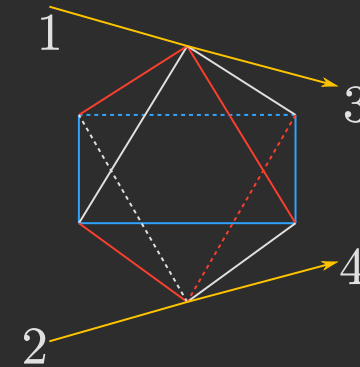
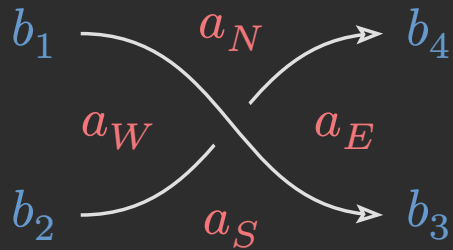
Four tetrahedra around a new central edge

FIVE-TERM OCTAHEDRON



Four outer tetrahedra and a central tetrahedron τ_0

SHAPE PARAMETERS AND OCTAHEDRAL COORDINATES



Vertical edges / horizontal edges

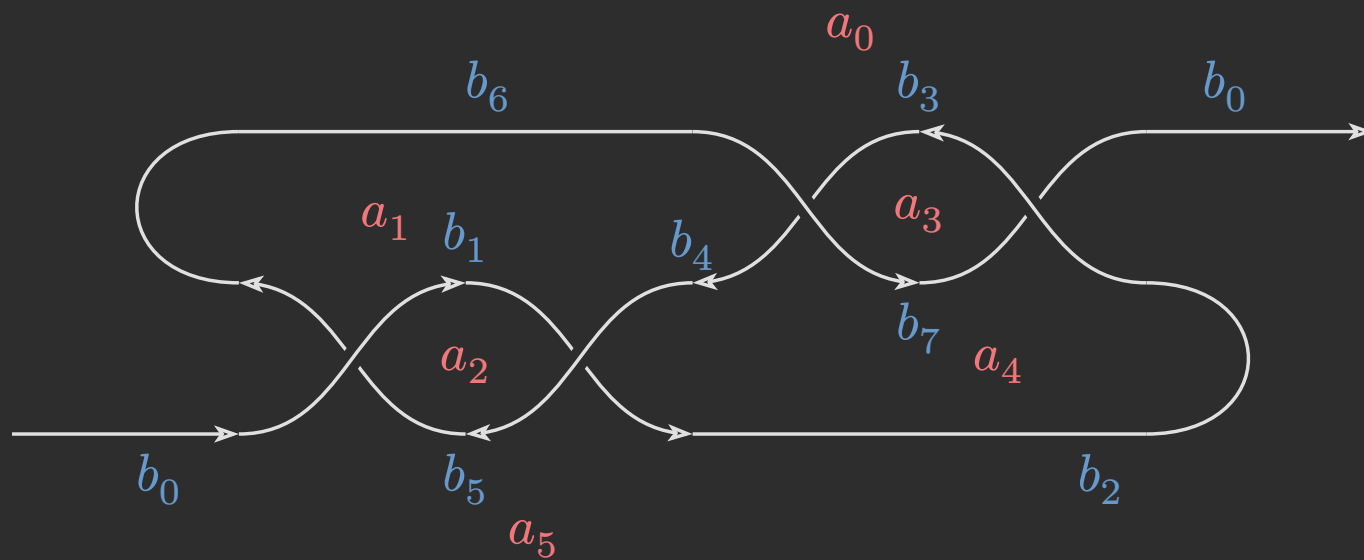
Complex angles of **horizontal edges** are related to b_i and m

Complex angles of **vertical edges** are related to a_j and m

Octahedral coordinates assign $b_i \in \mathbb{C}^\times$ to segments and $a_j \in \mathbb{C}^\times$ to regions, with meridian parameter m . Need to satisfy relations like

$$\frac{1 - b_2/(mb_1)}{1 - b_4/b_1} = \frac{a_W}{ma_N}.$$

EXAMPLE: FIGURE EIGHT



Proposition Choosing octahedral coordinates give a holonomy representation $\rho : \pi \rightarrow \mathrm{PSL}_2(\mathbb{C})$, with an $\mathrm{SL}_2(\mathbb{C})$ lift carrying the prescribed meridian parameter m .

Proof idea Work out how to encode ρ in terms of face-pairings, use these to assign ideal points, then check the gluing equations against the combinatorics of the diagram.

Question How do we recover coordinates from ρ ?

DECORATIONS AND SHADOW COLORINGS

A **decoration** chooses a distinguished eigenline for the peripheral subgroup of $\rho : \pi \rightarrow \mathrm{SL}_2(\mathbb{C})$.

For a knot:

- A parabolic meridian has one eigenline.
- A meridian with distinct eigenvalues has two.
- A scalar meridian gives $\mathbb{C}P^1$ -many choices.

This choice is extra geometric data!

Use row eigenvectors with

$$v_i g_i = m^{-1} v_i, \quad g_i = \rho(x_i).$$

(Sorry about the -1 here!)

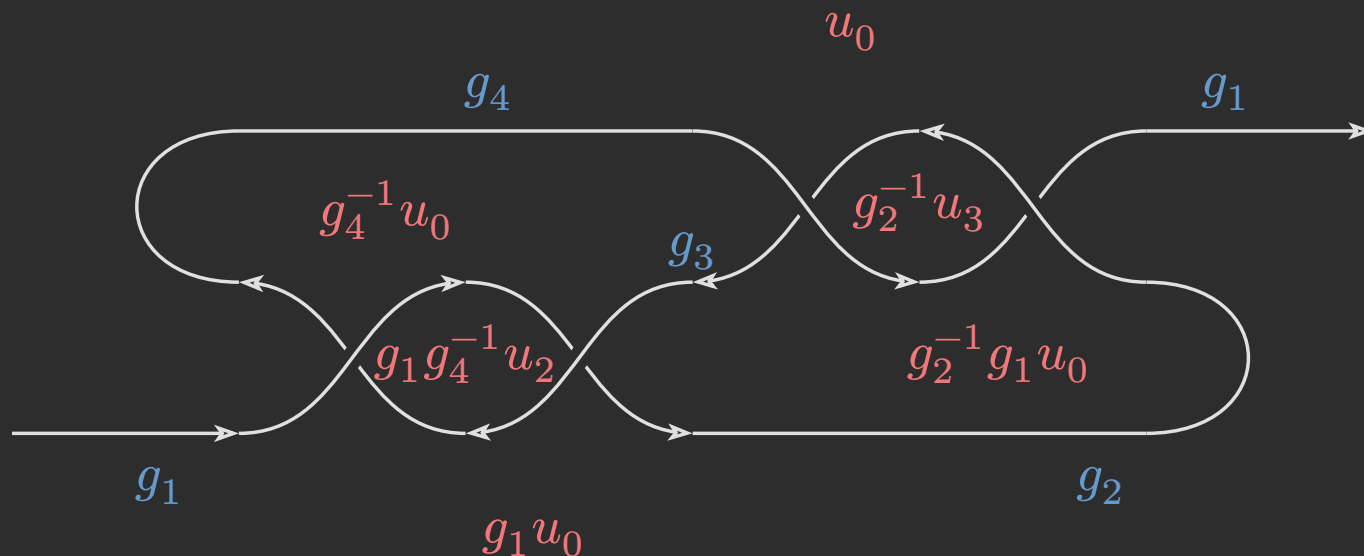
A **shadow coloring** assigns nonzero column vectors $u_j \in \mathbb{C}^2$ to regions:

$$\begin{array}{ccc} & u_j & \\ & \longrightarrow & g_i \\ & u_k = g_i u_j & \end{array}$$

If we choose the shadow of one region the rule determines the rest.

Geometrically these correspond to the arbitrary locations of the extra ideal points in $\mathbb{C}P^1$. This can always be changed by conjugation/gauge transformation.

EXAMPLE: FIGURE EIGHT



Here $g_i = (g_i, L_i)$ is a **decorated matrix**, an element of $\mathrm{SL}_2(\mathbb{C})$ plus eigenline $L_i = \mathrm{span}(v_i)$ (This is an augmentation of the conjugation quandle of $\mathrm{SL}_2(\mathbb{C})$.)

Wirtinger relations are things like $g_4 = g_1^{-1}g_3g_1, L_4 = L_3g_1$

Theorem 1 For an **admissible** decorated shadow coloring, set

$$a_j = \det(u_j, e_2), \quad b_i = -\frac{v_i e_2}{v_i u_{\text{above}(i)}},$$

where $e_2 = (0, 1)^T$.

This gives an octahedral coloring with holonomy ρ .

Admissibility: all a_j, b_i are finite and nonzero. This is always possible after conjugation, but might give a geometrically degenerate shape parameter.

Simplest proof [Cho, 2018]; [Inoue and Kabaya, 2014];

[Yoon, 2021]

Think about the geometry of ideal octahedra, keep track of ideal points.

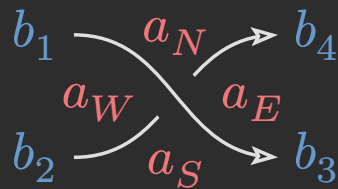
My proof [McPhail-Snyder, 2022]; [McPhail-Snyder, 2025]

I was thinking about representations of $\mathcal{U}_q(\mathfrak{sl}_2)$ at a root of unity. Went on a rather long detour (**biquandle factorizations** [Blanchet, Geer, Patureau-Mirand, and Reshetikhin, 2020]) before I realized octahedra were the right thing.

Better (?) proof Use cluster coordinates on framed, pinned flat connections of [Goncharov and Shen, 2019]

If we can find coordinates for ρ , are the tetrahedra
nondegenerate?

RECOVERING SHAPES OF TETRAHEDRA



$\mathcal{O}_D^4$ shapes

$$z_N = \frac{b_4}{b_1}, \quad z_W = \frac{b_2}{mb_1},$$

$$z_S = \frac{b_2}{b_3}, \quad z_E = \frac{mb_4}{b_3}.$$

$\mathcal{O}_D^5$ shapes

$$z_1^0 = \frac{a_W}{ma_N}, \quad z_2^0 = \frac{a_W}{ma_S},$$

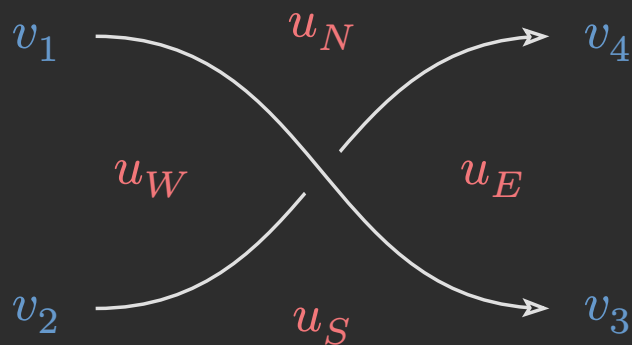
$$z_3^0 = \frac{ma_E}{a_S}, \quad z_4^0 = \frac{ma_E}{a_N},$$

$$z_0^0 = \frac{a_N a_S}{a_W a_E}.$$

Recall: a tetrahedron is nondegenerate when $z^0 \notin \{0, 1, \infty\}$.

SHAPES FROM THE SHADOW COLORING

Here u_N, u_W, u_S, u_E are the region shadows and $e_2 = (0, 1)^T$.



Generically can move around so no z_j^0 is 1.

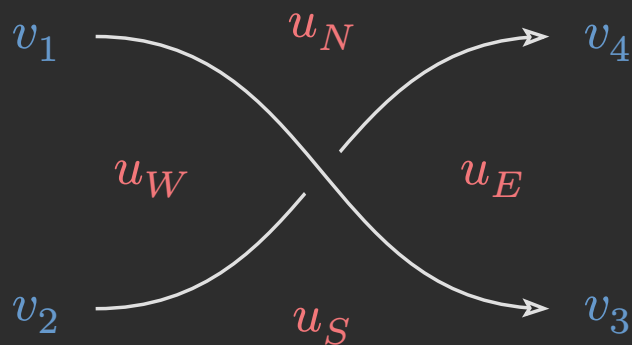
$\mathcal{O}_D^5$ shapes

$$\begin{aligned} z_1^0 &= \frac{\langle u_W, e_2 \rangle}{m \langle u_N, e_2 \rangle} = \frac{\langle u_W, e_2 \rangle}{\langle u_N, e_2 \rangle} \frac{\langle u_W, \hat{v}_1 \rangle}{\langle u_N, \hat{v}_1 \rangle}, \\ z_2^0 &= \frac{\langle u_W, e_2 \rangle}{m \langle u_S, e_2 \rangle} = \frac{\langle u_W, e_2 \rangle}{\langle u_S, e_2 \rangle} \frac{\langle u_S, \hat{v}_2 \rangle}{\langle u_W, \hat{v}_2 \rangle}, \\ z_3^0 &= \frac{m \langle u_E, e_2 \rangle}{\langle u_S, e_2 \rangle} = \frac{\langle u_E, e_2 \rangle}{\langle u_S, e_2 \rangle} \frac{\langle u_E, \hat{v}_3 \rangle}{\langle u_S, \hat{v}_3 \rangle}, \\ z_4^0 &= \frac{m \langle u_E, e_2 \rangle}{\langle u_N, e_2 \rangle} = \frac{\langle u_E, e_2 \rangle}{\langle u_N, e_2 \rangle} \frac{\langle u_N, \hat{v}_4 \rangle}{\langle u_E, \hat{v}_4 \rangle}, \\ z_0^0 &= \frac{\langle u_N, e_2 \rangle \langle u_S, e_2 \rangle}{\langle u_W, e_2 \rangle \langle u_E, e_2 \rangle}. \end{aligned}$$

The symplectic dual of the row $v = (v^1, v^2)$ is the column $\hat{v} = (-v^2, v^1)^T$.

With $\langle u, w \rangle = \det(u, w)$, we have $\langle u, \hat{v} \rangle = vu$.

SHAPES FROM THE SHADOW COLORING



If two adjacent v_i, v_j are linearly dependent
the shape parameter is 1!

$\mathcal{O}_D^4$ shapes

$$z_N = \frac{(v_4 e_2)(v_1 u_N)}{(v_4 u_N)(v_1 e_2)} = \frac{[v_1 \varphi(u_N)]}{[v_4 \varphi(u_N)]}$$

$$z_W = \frac{(v_2 e_2)(v_1 u_N)}{m(v_2 u_W)(v_1 e_2)} = \frac{[v_1 \varphi(u_W)]}{[v_2 \varphi(u_W)]}$$

$$z_S = \frac{(v_2 e_2)(v_3 u_E)}{(v_2 u_W)(v_3 e_2)} = \frac{[v_3 \varphi(u_W)]}{[v_2 \varphi(u_W)]}$$

$$z_E = \frac{m(v_4 e_2)(v_3 u_E)}{(v_4 u_N)(v_3 e_2)} = \frac{[v_3 \varphi(u_W)]}{[v_4 \varphi(u_W)]}$$

where $[(a, b)] = \frac{a}{b}$ and $\varphi(u) = (u \ e_2)$

EXISTENCE THEOREM

Theorem 2 (five-term) [Cho, 2016]; [Yoon, 2021]

If ρ sends no meridian to $\pm I$, there is a choice of gauge and shadow for which tetrahedron of $\mathcal{O}_D^5$ is nondegenerate.

(Remember: we do not require positivity of the shapes.)

Theorem 2 (four-term) [Me] [McPhail-Snyder, 2025]

A decorated ρ is detected by $\mathcal{O}_D^4$ if and only if the two distinguished eigenlines at each crossing are distinct:

$$L_1 \neq L_2.$$

If $L_1 = L_2$ we call the crossing **pinched** [Kim, Kim, and Yoon, 2018]. It is a geometric property: we can not avoid it by gauge transformation.

WHEN IS $\mathcal{O}_D^4$ ESSENTIAL?

A triangulation is **essential** if no edge is homotopic into the boundary.

A nondegenerate solution of the gluing equations implies essentiality.

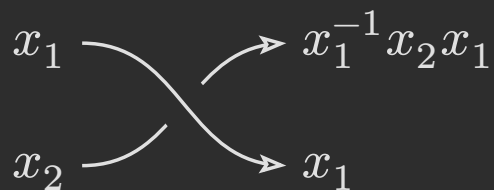
Corollary If some decorated representation is not pinched, then $\mathcal{O}_D^4$ is essential.

This condition is easy to check in practice. The **symplectic quandle** parametrizes parabolic elements of $\mathrm{SL}_2(\mathbb{C})$ by their fixed points, and <https://diagram.site> [Kim and Choi, 2025] is a database of these representations.

There is also a version for boundary non-parabolic representations.

A GROUP-THEORETIC VERSION

A diagram D is **arc-faithful** if at every crossing



we have $x_1 \neq x_2$ in π .

Theorem 3 [Me] [McPhail-Snyder, 2025]

For a hyperbolic knot, these are equivalent:

1. D is arc-faithful
2. ρ_{hyp} is detected by $\mathcal{O}_D^4$

Proof

If D is not arc-faithful any representation will be pinched.

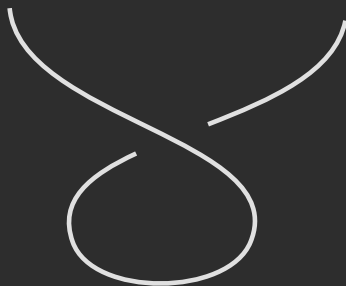
Conversely, suppose $\rho_{\text{hyp}}(x_1), \rho_{\text{hyp}}(x_2)$ share a fixed point p . Meridians are conjugate so $x_2 = yx_1y^{-1}$.

Since parabolics have a single fixed point $\rho_{\text{hyp}}(y)p = p$, so $\rho_{\text{hyp}}(y)$ also commutes with $\rho_{\text{hyp}}(x_1)$ and $\rho_{\text{hyp}}(x_2)$ and thus

$$\rho_{\text{hyp}}(x_2) = g\rho_{\text{hyp}}(x_1)g^{-1} = \rho_{\text{hyp}}(x_1).$$

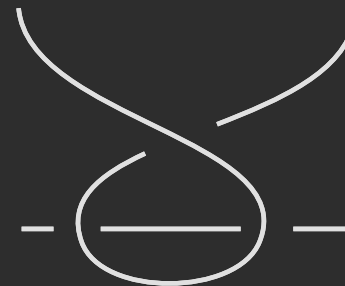
Faithfulness gives $x_2 = x_1$.

EXAMPLES



A kink

The over-arc and under-arc at this crossing have the same Wirtinger generator.



Nugatory crossings

Any nugatory crossing (like above) will not be arc-faithful.

Theorem [Garoufalidis, Moffatt, and Thurston, 2016]; [Sakuma and Yokota, 2018]

Reduced alternating diagrams of hyperbolic knots are arc-faithful. (Those sources proved they are essential, but that's the same thing!)

Conjecture Every hyperbolic knot has an arc-faithful diagram.

A proof I tried

Suppose D has a crossing with $x_1 = x_2$.

1. A local crossing disc B meets K twice, with boundary representing $x_1 x_2^{-1}$ in the complement.

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Seonhwa Kim mentioned to me last week he might have a proof using more involved geometric topology.

Christian Blanchet, Nathan Geer, Bertrand Patureau-Mirand, and Nicolai Reshetikhin. *Holonomy braidings, biquandles and quantum invariants of links with $SL_2(\mathbb{C})$ flat connections* (2020). Selecta Mathematica. doi:10.1007/s00029-020-0545-0

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Yoshiyuki Yokota. *On the volume conjecture for hyperbolic knots* (2000). <https://arxiv.org/abs/math/0009165>

Seokbeom Yoon. *On the potential functions for a link diagram* (2021). doi:10.1142/S0218216521500565. <https://arxiv.org/abs/1810.09080>