Complex Chern-Simons invariants via the Reshetikhin-Turaev construction

Abstract: The Chern-Simons invariant $\mathcal{J}$ of a flat $\mathfrak{sl}_2$ connection on a 3-manifold (equivalently, a generalized hyperbolic structure) is an important geometric invariant that can be understood as a complexification of the hyperbolic volume. Because it is defined using an integral it is natural to expect it can be computed via cutting and gluing, as in a TQFT; this is well-understood for ideal triangulations. In this talk I will explain how to compute $\mathcal{J}$ for tangle exteriors using the Reshetikhin-Turaev formalism and how this naturally leads to a quantization of $\mathcal{J}$ that generalizes Kashaev's invariant. This perspective suggests the existence of a $\mathrm{SL}_2(\mathbb{C})$-crossed braided tensor category capturing $\mathcal{J}$ and its quantization, and I will indicate what is known and what remains to be worked out. Parts of this talk are based on joint work with N. Reshetikhin.

Plan

1. What is the Chern-Simons invariant?
2. How to compute it from a tangle diagram
3. Connections to tensor categories
4. Quantization (if time)

1 What is the Chern-Simons invariant?

1.1 Integrals, closed manifolds

1.1a Definition

- M closed 3-manifold
- $\rho : M \to \mathrm{SL}_2(\mathbb{C})$ lift of a generalized hyperbolic structure.
  - $\mathrm{Isom}(\mathbb{H}^3) \cong \mathrm{PSL}_2(\mathbb{C})$
  - Choose a flat connection A with holonomy ρ
- $$S(M, A) = \frac{i}{8\pi i} \int_M \mathrm{tr}[A, \mathrm{d}A] + \frac{2}{3} \mathrm{tr}[A, A \wedge A]$$

  is independent of the gauge class of A modulo $2\pi i\mathbb{Z}$.
- We use multiplicative version $\mathcal{J}(M, \rho) = e^{S(M,A)}$.

1.1b Geometric meaning

- If ρ induces a hyperbolic metric g on M then by a theorem of Yoshida,

  $$S(M, A) = \mathrm{Vol}(M, g) + i \, \mathrm{CS}(M, g)$$

  where the imaginary part is the Chern-Simons invariant of the $\mathrm{SO}(3)$ frame-field of g.
  - Hyperbolic volume is a strong invariant!
- Get a function $\mathcal{J} : R(M) \to \mathbb{C}$ on the $\mathrm{SL}_2(\mathbb{C})$ representation variety of M.
  - It's gauge-invariant, so could say it's a function "on the character variety"
  - When M is hyperbolic there is a special distinguished ρ_{hyp} but there are also other representations

1.2 Local computations

- $\mathcal{J}$ is defined by an integral, so it should admit a cut-and-paste/TQFT description
- The Chern-Simons invariant of a manifold with boundary is a section of a line bundle over the moduli space of flat connections
- What does this actually mean?

1.2a Concrete example: tetrahedra

- Consider an ideal tetrahedron τ (vertices at infinity in $\mathbb{H}^3$)
- Geometry of τ is described by cross-ratio of vertices, a **shape parameter** $z \in \mathbb{C}P^1 \setminus \{0, 1, \infty\}$
  - For a manifold composed of these, shapes will to satisfy **gluing equations** at each edge class
- z is not enough to compute $S(\tau)$!
  - Integral is related to the **dilogarithm**

$$\mathrm{Li}_2(z) = \int_0^z \frac{-\log(1-t)}{t}\,\mathrm{d}t$$

  which has a branch point at 1. Crossing the cut picks up a $\log(z)$ which has a branch point at 0. We need to consider a related covering space

  - more geometrically, need to fix boundary conditions for the flat connection A
- call this data a **flattening** (ζ^0, ζ^1), branches of $\log(z)$ and $-\log(1-z)$
  - Now value is given by a lift of the dilogarithm

$$S(\tau, \zeta^0, \zeta^1) = \frac{\mathrm{Li}_2\left(e^{2\pi i \zeta^0}\right)}{2\pi i} + \zeta^0 \log\left(1 - e^{2\pi i \zeta^0}\right) + \pi i \zeta^0 \zeta^1 - \frac{\pi i}{12}$$

  - Depends on the flattening in a simple way, for example

$$S(\tau, \zeta^0 + 1, \zeta^1) \equiv S(\tau, \zeta^0, \zeta^1) - \pi i \zeta^1 \bmod 2\pi i \mathbb{Z}$$
$$S(\tau, \zeta^0 1, \zeta^1 + 1) \equiv S(\tau, \zeta^0, \zeta^1) + \pi i \zeta^1 \bmod 2\pi i \mathbb{Z}$$

- Related to a line bundle over $\mathbb{C}P^1 \setminus \{0, 1, \infty\} =$ moduli space of vertex-ordered hyperbolic ideal tetrahedra.
  - Fiber over z is the set of triples (x, ζ^0, ζ^1) with

$$e^{2\pi i \zeta^0} = z \quad e^{2\pi i \zeta^1} = \frac{1}{1-z}$$

  and first argument transforming as above

- When we assemble a manifold from tetrahedra, the dependence on logarithms will cancel out and we get a number.

  - To do this, we need to make sure the choice of logarithms is coherent. It is true (but not obvious) that one exists for a shaped triangulation

2 $\mathcal{J}$ as a Reshetikhin-Turaev invariant

Triangulations are nice, but sometimes tangle diagrams are better.

2.1 Goal

- Category $\mathcal{T}$ of tangles with geometric data
- Category $\mathcal{V}$ of vector spaces
- Monoidal functor $\mathcal{J}^{\psi} : \mathcal{T} \to \mathcal{V}$.
  - ψ indicates a change in normalization
  - more convenient, differs by an elementary term
- $\mathcal{J}^{\psi}$ recovers usual meanings of Chern-Simons invariant of 3-manifolds and links (details later, or if someone asks)
  - Similar computational results for links and diagrams are known [Inoue, Cho, et al.]
- The value on a tangle (not a whole link) will also be an invariant
  - To my knowledge this is a new result

2.2 The tangle category

2.2a First attempt

- Category of oriented, framed tangles
- Label tangles by values of ρ on Wirtinger generators (meridians)
- Add eigenspaces to capture decorations
  - these are a natural choice of extra data for torus boundary components
  - related to "flattenings" in cluster algebras literature
- An example, including a crossing in detail

2.2b Problem

- Need to keep track of basepoints!
- Natural framework for tetrahedra is fundamental groupoid
  - one basepoint for each τ
  - morphisms paths across faces

  - we are using the Wirtinger presentation, where there is only one basepoint
- Concretely, hard to extract shape parameters directly from labeling data
- To fix, use **shadow coloring** with labellings of regions by elements of $\mathbb{C}^2$
  - add in the shadow colors
- Now $\mathcal{T}$ is the category of tangles with decorated representations *and* shadow colorings
  - No longer a monoidal category, but a 2-category

2.2c The functor on crossings

- Let D be a diagram of a tangle T. The **octahedral decomposition** is a (semi)-ideal triangulation of the complement of T determined by D.
  - Each crossing gets four tetrahedra
  - One for each corner, gluing along edges and regions
  - How can we extract their geometric data (shape parameters)?
  - What about flattenings?

- Using the shadow colorings each segment gets a parameter

$$b_i = -\frac{v_i \boldsymbol{e}_2}{v_i u_i}$$

  where u_i is above the segment
  - Shape parameters are given by ratios of the b_i and m_i label a crossing with some

- How do we get a flattening? I.e., how do we choose boundary conditions for the flat connection on each tetrahedron?
  - By using normalization $\mathcal{J}^{\psi}$ only need ζ^0 (not ζ_1)
  - To choose them, choose logarithms β_i of the b_i
  - These will automatically satisfy the coherence condition (they are a version of Zickert's **Ptolemy coordinates**) [cite]

- now we assign a (positive) crossing the number $e^{\Omega(\cdot)}$ where

$$\Omega(\mu_1, \mu_2 |\ \beta_1, \beta_2, \beta_3, \beta_4) = \exp\left[L(\zeta_N^0) + L(\zeta_S^0) - L(\zeta_E^0) - L(\zeta_W^0)\right]$$

  - here L is another dilog variant

$$L(\zeta^0) = \frac{\mathrm{Li}_2\left(e^{2\pi i\zeta^0}\right)}{2\pi i} + \zeta^0 \log\left(1 - e^{2\pi i\zeta}\right) - \frac{\pi i}{12}$$

- What kind of object is this? Not a function of coloring, since it depends the choice of β_i
- Again, there is a simple transformation rule:

$$e^{\Omega(\beta_1+1)} = a_1 e^{\Omega(\beta_1)}$$

where

$$a_1 = \frac{\boldsymbol{e}_1 u}{\boldsymbol{e}_1 u'}$$

is determined using **region parameters** derived from shadow coloring

2.2d What kind of object do we have?

- At a crossing, if each segment is assigned a complex line with basis vectors

$$|\beta_i + 1\rangle = a_i |\beta_i\rangle$$

corresponding to choices of logarithms of segment parameters

- Now the map

$$|\beta_1\rangle \otimes |\beta_2\rangle \to e^{\Omega(\cdots)} |\beta_4\rangle \otimes |\beta_3\rangle$$

is well-defined and independent of choices!

2.2e Statement of the theorem

- $\mathcal{T}$ is the 2-category of oriented framed tangle diagrams with decorated representations and shadow colorings that are admissible.[1] We also choose **log-meridians** μ_i globally for each component
- $\mathrm{B}\mathcal{V}$ is the delooping of the monoidal category of $\mathbb{C}$ vector spaces
- We define a 2-functor $\mathcal{J}^{\psi} : \mathcal{T} \to \mathrm{B}\mathcal{V}$

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[1] If the a_i or b_i are 0 or ∞ then our functions are not well-defined. Up to gauge transformation everything is admissible.

Objects each $u \in \mathbb{C}^2$ assigned the unique object of $\mathrm{B}\mathcal{V}$

1-morphisms $(g, [v]) : u \to gu$ assigned a complex line with a basis $\{|\beta\rangle\}$ for each logarithm of its segment parameter. Different values are proportional according to ratio of adjacent region parameters:

$$|\beta_i + 1\rangle = a_i |\beta_i\rangle$$

2-morphisms decompose into crossings, cups and caps
  - crossings get e^{Ω} as above
  - cups and caps are interpreted using pivotal structure of $\mathcal{V}$

- **Theorem** [Me] $\mathcal{J}^{\psi}$ depends only on the isotopy class of the tangle diagram and recovers the usual Chern-Simons invariant
  - value on a link and a boundary-parabolic representation is usual Chern-Simons invariant
  - value on a framed link and rep admitting Dehn filling is the invariant of the filled manifold
- The dependence on the β_i is captured by the vector spaces assigned to boundary points. What about the μ_i?
- Say K is a knot. Then

$$\mathcal{J}^{\psi}(K, \rho, \mu + 1) = \ell(\rho)^2 \mathcal{J}^{\psi}(K, \rho, \mu)$$

where $\ell(\rho)$ is the longitude eigenvalue distinguished by the orientation, framing, and decoration

2.3 What have we obtained?

- The values of $\mathcal{J}^{\psi}$ on points are naturally viewed as fibers of a line bundle over the representation variety of a point
- Similarly, the values on tangles are sections of a line bundle over the representation variety of the tangle

3 Towards a crossed braided category

3.1 Did we actually generalize the RT construction?

We have a *functor* $\mathcal{J}^{\psi}$ from a tangle category to an algebraic category that gives tangle invariants.

- You could say this is "Reshetikhin-Turaev" but we have not quite done that
- RT showed that given a monoidal category $\mathcal{C}$ with the right structure (a ribbon structure) there is *automatically* a functor from $\mathcal{C}$-labeled tangles to $\mathcal{C}$
- We have instead constructed such a functor "by hand"
  - Could think of it as a "representation" or "model" of $\mathcal{T}$ valued in $\mathrm{B}\mathcal{V}$

3.2 Gradings and crossed structure

- Set $G = \mathrm{SL}_2(\mathbb{C})$, $X = \mathbb{C}^2$ a G-module
- Call our hypothetical 2-category $\mathcal{C}$

Objects graded by X (in fact, equal to in some sense)

1-morphisms graded by X and G. Something in degree u, g is a map $u \to gu$

2-morphisms preserve the gradings

- The braiding on this category is crossed for both gradings
  - It is also not quite a morphism, but a section of a bundle of morphisms (because of the μ dependence)
  - 1-morphisms are independent of μ_i

3.3 Questions

1. What is the right definition of crossed braiding here?
   - Is it related to gradings things by homotopy 2-types instead of groups?
2. What is the right description of the dependence of morphisms on the μ_i?
3. Can this braiding be descried in a more systematic way?
   - All Reidemeister moves have to be checked by hand
   - If we had functoriality of the braiding this would be eaiser

4 Quantization

I don't really have time to discuss properly, but one can define a quantization $\mathcal{Z}_N^{\psi}$

- Fix integer N, quantization parameter is $\omega = e^{2\pi i N}$

- The thing assigned to a point is now a N-dimensional vector space
  - Instead of $|\beta + 1\rangle = a|\beta\rangle$ we do $|\beta + N\rangle = a|\beta\rangle$
  - These become cyclic modules for the large quantum group $\mathcal{U}_\omega(\mathfrak{sl}_2)$
- The braiding kernel is now the matrix of a linear map

Theorem [Me, N. Reshetikhin] The braiding matrices can be derived from the quasitriangular structure on the quantum group.

When we first proved this theorem we knew they were well-defined but did not know how to interpret them. Making analogies to $\mathcal{J}^\psi$ made everything work!

Theorem [Me] This construction gives a well-defined invariant of tangles.

- It should be understood as a quantization of $\mathcal{J}^\psi$.
- When ρ is trivial we recover Kashaev's invariant.
- What about log-meridians? Say we have a knot K.
  - There is a similar N-periodicity for μ

    $$\mathcal{Z}^\psi_{N(K,\rho,\mu+N)} = \ell(\rho)^{-2}\mathcal{Z}^\psi_{N(K,\rho,\mu)}$$
  - together the N values

    $$\mathcal{Z}^\psi_{N(K,\rho,\mu)}, \dots, \mathcal{Z}^\psi_{N(K,\rho,\mu+N-1)}$$

  give a vector-valued invariant depending only on ρ, so we view $\mathcal{Z}^\psi_N$ as a section of a *vector* bundle.
- Similar vector bundles of quasi-periodic functions occur in geometric quantization of complex Chern-Simons theory.
- Just like for $\mathcal{J}^\psi$ it would be interesting to understand the algebraic structure of the target (2-)category of $\mathcal{Z}^\psi$. Here we already know it will not be semisimple.