Notes on quantum holonomy invariants

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Abstract

These are some informal notes on quantum holonomy invariants. They are intended for someone who knows something about hyperbolic knot theory and ideal triangulations but less about representation theory and quantum topology.

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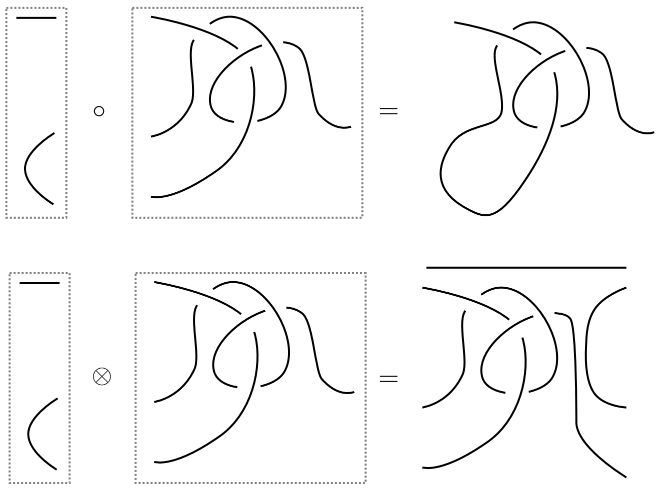

Figure 2: Tangle composition $T_2 \circ T_1$ and tensor product $T_1 \otimes T_2$. Here T_2 is the more complicated tangle. Because function composition is read left-to-right we need to either take an unnatural convention on $\circ$ or write function composition right-to-left; here we have chosen the former.

1. Shaped tangles

For a more detailed discussion of these topics, see [McP22].

[McP22] C. McPhail-Snyder, *Hyperbolic structures on link complements, octahedral decompositions, and quantum $\mathfrak{sl}_2$*. arXiv

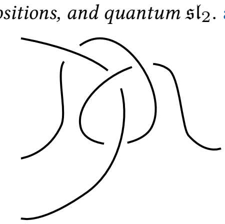

Figure 1: A tangle.

Definition 1.1. A *link* is an embedding of a disjoint union of circles into S^3 up to isotopy. A *tangle* is similar, but we embed into $[0,1] \times \mathbb{R}^2$ and allow intervals whose endpoints are mapped to the boundary. For example, Figure 1 is an embedding of two intervals. As usual we represent tangles by diagrams, which are like link diagrams but with boundaries.

One reason to consider tangles is that we can compose them together into other tangles or links. There are two ways to do this: we can place two tangles in parallel, or we can glue tangles together if the endpoints match; see Figure 2. Formally, we say that tangles are a *monoidal category*; we will discuss this more later.

Let T be a tangle. The complement of a tangle is similar to the complement of a link, but the complement is taken in $[0,1] \times \mathbb{R}^2$. We write $\pi(T)$ for the fundamental group of the tangle complement. A $\mathrm{SL}_2(\mathbb{C})$*-structure* is a representation $\rho : \pi(T) \to \mathrm{SL}_2(\mathbb{C})$, where

$$\mathrm{SL}_2(\mathbb{C}) = \{A \in M_{2\times 2}(\mathbb{C}) \mid \det A = 1\}$$

is the group of 2×2 complex matrices with determinant 1. We can think of ρ as a generalized hyperbolic structure, because $\mathrm{SL}_2(\mathbb{C})$ double-covers $\mathrm{PSL}_2(\mathbb{C}) = \mathrm{SL}_2(\mathbb{C})/\{\pm I\}$, which can be identified with the isometry group $\mathrm{Isom}(\mathbb{H}^3)$ of hyperbolic 3-space.[1] We want to describe $\mathrm{SL}_2(\mathbb{C})$-structures in terms of tangle diagrams. Here is one way:

Definition 1.2. An *arc* of a tangle diagram is broken at undercrossings, but not overcrossings. The *Wirtinger presentation* of $\pi(T)$ has one generator for each arc and one relation for each crossing. To make this consistent we need to *orient* T so that we can tell w and w^{-1} apart. Topologically, we are putting the basepoint at the top of the diagram, and the generator w assigned to an arc a comes from a path that travels above the diagram to near a, wraps once

[1] Isometries of $\mathbb{H}^3$ act on the sphere at infinity, which we identify with the Riemann sphere $\mathbb{C} \cup \{\infty\}$. It is known that the group of automorphisms of the Riemann sphere is $\mathrm{PSL}_2(\mathbb{C})$ acting by fractional linear transformations.

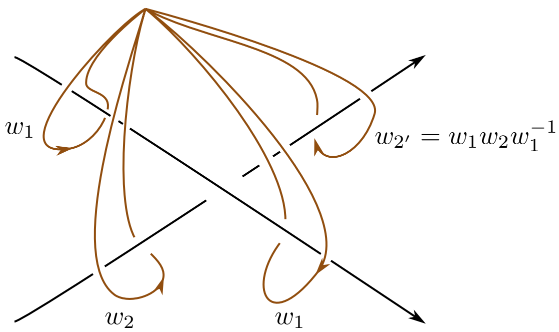

Figure 4: Deriving relations for the Wirtinger generators at a crossing.

in a positive direction (according to the orientation of a) around it, then travels back to the basepoint above the diagram.

EXAMPLE 1.3. In Figure 3 we have a diagram of the trefoil knot with three arcs. The corresponding presentation is

$$\langle w_1, w_2, w_3 | w_3 w_1 = w_2 w_1, w_1 w_2 = w_2 w_3, w_2 w_3 = w_3 w_1 \rangle .$$

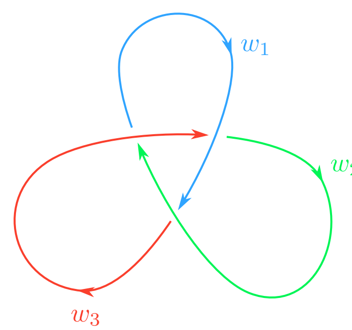

Figure 3: The three arcs in the usual diagram of the trefoil knot and the corresponding generators w_1, w_2, w_3.

The generators of the Wirtinger presentation are *meridians*. Notice that all meridians of a connected component are conjugate. More generally, if we focus on a single crossing we can write the meridians of the outgoing arcs in terms of the incoming arcs, as in Figure 4.

DEFINITION 1.4. A $\mathrm{SL}_2(\mathbb{C})$*-coloring* of a tangle diagram D is a labeling of the arcs of D by $g_j \in \mathrm{SL}_2(\mathbb{C})$ so that $g_{2'} = g_1 g_2 g_1^{-1}$ at each positive crossing (labeled as in Figure 4).

This makes sense for any group G. More generally, we can use an algebraic structure called a *quandle*, with an operation $*$ generalizing conjugation in a group. In this case we are using the conjugation quandle of $\mathrm{SL}_2(\mathbb{C})$, which has $g_1 * g_2 := g_1 g_2 g_1^{-1}$.

THEOREM 1.5. Let D be any diagram of a tangle T. There is a bijection between the set of representations $\rho : \pi(T) \to \mathrm{SL}_2(\mathbb{C})$ (i.e. the set of $\mathrm{SL}_2(\mathbb{C})$-structures) and the set of $\mathrm{SL}_2(\mathbb{C})$-colorings of D. $\lozenge$

Proof. This is basically the same thing as claiming that the Wirtinger presentation really is a presentation of $\pi(T)$, which is well-known. $\square$

The Wirtinger presentation is simple to describe, but is inconvenient from the point of view of hyperbolic geometry. In addition, it is incompatible with quantum group representation theory, as we will see later. We now want to describe a *different* way of representing $\mathrm{SL}_2(\mathbb{C})$-structures. It is more complicated, but it turns out to be much better geometrically.

DEFINITION 1.6. Let D be an oriented tangle diagram. The *segments* of D are like arcs but are broken at both overcrossings and undercrossings. (It is common to call these “edges” but we follow the terminology of [KKY18] to avoid confusion with edges of ideal polyhedra.) The *regions* of D are the connected components of the plane minus D. The *fundamental groupoid* $\Pi(D)$ has

OBJECTS for each region of D

MORPHISMS generated by a path $x_i^{\pm}$ above and a path below each segment, as in Figure 5.

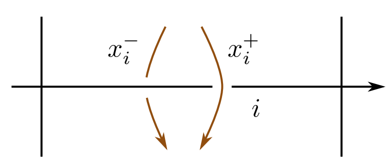

Figure 5: The two arcs of $\Pi(D)$ at a segment.

[KKY18] H. Kim, S. Kim, and S. Yoon, “Octahedral developing of knot complement. I: Pseudo-hyperbolic structure”. arXiv DOI

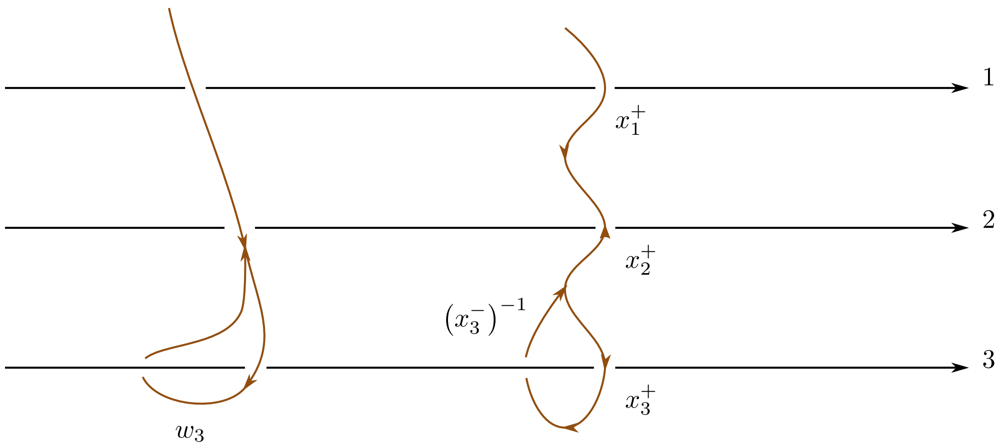

Figure 7: The path w_3 in $\pi(D)$ and the path $x_1^+ x_2^+ x_3^+ \left(x_3^-\right)^{-1} \left(x_2^+\right)^{-1} \left(x_1^+\right)^{-1}$ in $\Pi(D)$ are equivalent.

There are relations between the paths at each crossing:

$$x_1^- x_2^- = x_{2'}^- x_{1'}^- \qquad x_1^- x_2^+ = x_{2'}^+ x_{1'}^- \qquad x_1^+ x_2^+ = x_{2'}^+ x_{1'}^+$$

Recall that a fundamental groupoid is like a fundamental group, but with more than one basepoint. Two paths might not be composable if they do not start in the same place. To understand where the relations come from, look at Figure 6.

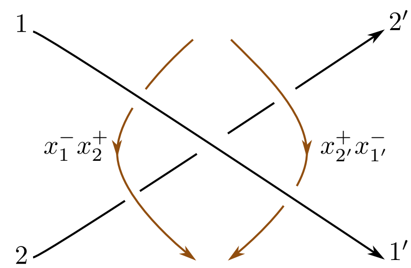

Figure 6: Deriving the middle relation of $\Pi(D)$ at a crossing.

Proposition 1.7. $\Pi(D)$ is equivalent to $\pi(D)$ (the usual fundamental group). ◊

Proof. We can write the Writinger generators in terms of the groupoid generators, as in Figure 7. □

By this proposition we can describe $\mathrm{SL}_2(\mathbb{C})$-structures by giving representations $\Pi(D) \to \mathrm{SL}_2(\mathbb{C})$. We will do something slightly more general: we will describe a map $\Pi(D) \to \mathrm{GL}_2(\mathbb{C})$ in such a way that the Wirtinger generators have determinant 1 and thus lie in $\mathrm{SL}_2(\mathbb{C})$. Because morphisms of $\Pi(D)$ are associated to segments our description will be too.

Definition 1.8. A *shape* is a triple of nonzero complex numbers. We usually denote a shape by $\chi = (a, b, m) \in (\mathbb{C} \setminus \{0\})^3$, and when it is assigned to a segment j of a tangle diagram we denote it $\chi_j = (a_j, b_j, m_j)$. To a diagram D with shapes assigned to its segments, we define the *holonomy representation*

$$\rho : \Pi(D) \to \mathrm{SL}_2(\mathbb{C})$$

by

$$\rho\left(x^+\right) = \begin{bmatrix} a & 0 \\ (a - 1/m)/b & 1 \end{bmatrix}, \qquad \rho\left(x^-\right) = \begin{bmatrix} 1 & (a-m)b \\ 0 & a \end{bmatrix}, \tag{1}$$

where the generators $x^\pm$ are associated to a strand of D colored with the shape $\chi = (a, b, m)$.

Remark 1.9. In terms of $\Pi(D)$, the meridian around a strand with shape χ is conjugate to the matrix

$$\rho\left(x^+ (x^-)^{-1}\right) = \begin{bmatrix} a & -(a-m)b \\ (a - 1/m)/b & m + m^{-1} - a \end{bmatrix} \tag{2}$$

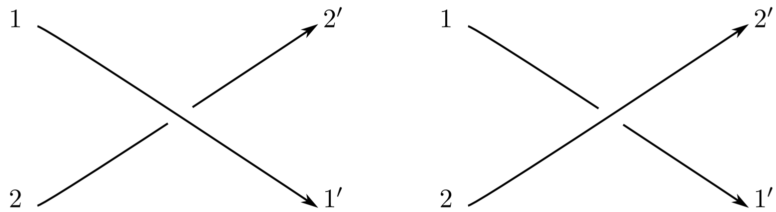

Figure 8: Positive (left) and negative (right) crossings.

which has trace $m + m^{-1}$. In general, the meridian is *not* equal to $\rho\left(x^+(x^-)^{-1}\right)$, as shown in Figure 7. However, this shows us that we can interpret the m-variable in a shape as an eigenvalue of the meridian around that strand.

Of course, we cannot assign any holonomies we want: there are relations in $\Pi(D)$ which need to be satisfied. To describe them we need a convention on types of crossings, as in Figure 8. We usually label the segments at a crossing by $1, 2, 1', 2'$ as in the figure.

Definition 1.10. The *braiding* B is the partially-defined map given by $B(\chi_1, \chi_2) = (\chi_{2'}, \chi_{1'})$, where

$$\begin{aligned} a_{1'} &= a_1 A^{-1} \\ a_{2'} &= a_2 A \\ A &= 1 - \frac{m_1 b_1}{b_2}\left(1 - \frac{a_1}{m_1}\right)\left(1 - \frac{1}{m_2 a_2}\right) \end{aligned} \tag{3}$$

$$\begin{aligned} b_{1'} &= \frac{m_2 b_2}{m_1}\left(1 - m_2 a_2\left(1 - \frac{b_2}{m_1 b_1}\right)\right)^{-1} \\ b_{2'} &= b_1\left(1 - \frac{m_1}{a_1}\left(1 - \frac{b_2}{m_1 b_1}\right)\right) \end{aligned} \tag{4}$$

$$m_{1'} = m_1 \quad m_{2'} = m_2 \tag{5}$$

We think of B as being associated to a positive crossing with incoming strands 1 and 2 and outgoing strands $2'$ and $1'$. A *shaping* of a tangle diagram D is an assignment of shapes to the segments of D, such that at every crossing we have

$$B(\chi_1, \chi_2) = (\chi_{2'}, \chi_{1'}) \text{ at positive crossings and}$$
$$B^{-1}(\chi_1, \chi_2) = (\chi_{2'}, \chi_{1'}) \text{ at negative crossings.}$$

See [McP22, Lemma 2.5] for a formula for B^{-1}.

Example 1.11. Consider the diagram of the trefoil in Figure 9 with the segments labeled by $1, \ldots, 6$. An assignment of shapes $\chi_i = (a_i, b_i, m_i), i = 1, \ldots, 6$ to the diagram is valid if

$$B(\chi_1, \chi_2) = (\chi_3, \chi_4),\ B(\chi_3, \chi_4) = (\chi_5, \chi_6), \text{ and } B(\chi_5, \chi_6) = (\chi_1, \chi_2).$$

This immediately implies that $m_1 = m_2 = \cdots = m_6 = m$; in general there is only one variable m for each component of the link.[2] A family of solutions is given by

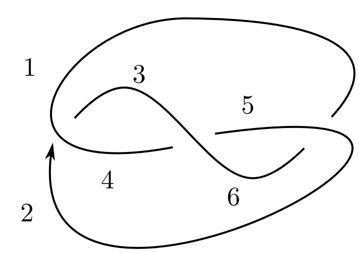

Figure 9: A diagram of the trefoil knot with labeled segments.

[2] Geometrically, m is an eigenvalue of the holonomy of a meridian, and there is one conjugacy class of meridian per link component.

$$\chi_1 = \left(\frac{b_1 m - b_2}{b_1 - b_3}, b_1, m\right)$$
$$\chi_2 = \left(-\frac{b_2^2 m^2 - b_2 b_3 m + b_1 b_3}{(b_1 m - b_2)(b_2 m - b_3)}, b_2, m\right)$$
$$\chi_3 = \left(-\frac{b_2 b_3 m^3 + b_1 b_3 m^2 - b_3^2 m^2 - b_1 b_2 m + b_1 b_3}{(b_2 m - b_3)(b_1 - b_3) m}, b_3, m\right)$$
$$\chi_4 = \left(\frac{\left(b_2^2 m^2 - b_2 b_3 m + b_1 b_3\right) m}{b_2 b_3 m^3 + b_1 b_3 m^2 - b_3^2 m^2 - b_1 b_2 m + b_1 b_3}, \frac{(b_2 m - b_3) b_1}{(b_2 m + b_1 - b_3) m}, m\right)$$
$$\chi_5 = \left(-\frac{b_2^2 m^4 - b_2 b_3 m^3 + b_2 b_3 m + b_1 b_3 - b_3^2}{(b_2 m - b_3)(b_1 - b_3) m}, \frac{b_1 b_2 m}{b_2 m + b_1 - b_3}, m\right)$$
$$\chi_6 = \left(\frac{\left(b_2^2 m^2 - b_2 b_3 m + b_1 b_3\right) m}{b_2^2 m^4 - b_2 b_3 m^3 + b_2 b_3 m + b_1 b_3 - b_3^2}, -\frac{b_1 b_3}{(b_2 m - b_3) m}, m\right)$$

where b_1, b_2, b_3 can be freely chosen as long as none of the a_i or b_i are 0 or ∞. It turns out that the choice of b_1, b_2, b_3 does not affect the conjugacy class of the $\mathrm{SL}_2(\mathbb{C})$-structure determined by this shaping.

In practice, the equations for all the a_i and b_i are usually difficult to solve. We can simplify them by either eliminating the b_i and solving them in terms of the a_i or vice-versa. The b_i are closely related to the segment variables of [KKY18], and in parallel the a_i are closely related to the region variables. For example, the solutions in the previous example were computed using the boundary-parabolic solution to the segment gluing equations of [KKY18, Example 6.7] to determine the b-variables, then using them to determine the a-variables.

Remark 1.12. The braiding B and the set of shapes X form a *biquandle*, more specifically a *generic biquandle* because B is only partially defined. What we have called a "shaping" of a diagram D is a coloring of D by this biquandle.[3] We refer to [Bla+20] for a discussion of quandles and biquandles in our context.

[3] In parallel a decoration of the arcs of D by elements of a group G satisfying Wirtinger relations at the crossings is a coloring by the conjugation quandle of G.

[Bla+20] C. Blanchet et al., "Holonomy braidings, biquandles and quantum invariants of links with $\mathsf{SL}_2(\mathbb{C})$ flat connections". arXiv DOI

Theorem 1.13. A shaping of a tangle diagram D gives a well-defined holonomy representation $\rho : \Pi(D) \to \mathrm{SL}_2(\mathbb{C})$, hence a well-defined representation $\pi(D) \to \mathrm{SL}_2(\mathbb{C})$ of the fundamental group. ♢

Proof. It suffices to check the relations between generators of $\Pi(D)$ at each crossing. It is tedious (but entirely elementary) to check that the braiding condition $B(\chi_1, \chi_2) = (\chi_{2'}, \chi_{1'})$ does this at positive crossings, and similarly at negative crossings. □

One might worry about the fact that B is only generically defined: there are certain singular pairs (χ_1, χ_2) for which $B(\chi_1, \chi_2)$ does not make sense, so we cannot put them on strands 1 and 2 of any positive crossing. Another issue is that not every $\rho : \pi(D) \to \mathrm{SL}_2(\mathbb{C})$ can be expressed in terms of the shapes. For example, matrices of the form

$$\rho\left(x^+ (x^-)^{-1}\right) = \begin{bmatrix} a & -(a-m)b \\ (a - 1/m)/b & m + m^{-1} - a \end{bmatrix} \text{ for } a, b, m \in \mathbb{C} \setminus \{0\}$$

are a proper subset of $\mathrm{SL}_2(\mathbb{C})$, so if the image of $x^+(x^-)^{-1}$ under some ρ lies outside of this subset then we cannot represent ρ using shapes. However, we are really only interested in ρ up to conjugacy, which turns out to be enough:

Theorem 1.14. Let D be any diagram of a tangle T. We say a $\mathrm{SL}_2(\mathbb{C})$-structure ρ of T (that is, a representation $\rho : \pi(T) \to \mathrm{SL}_2(\mathbb{C})$) is *detected* by D if there is a shaping of D with holonomy representation ρ. Every $\mathrm{SL}_2(\mathbb{C})$-structure on T is conjugate to one detected by D. ◊

This is one of the major results of [Bla+20].

Proof idea. At each crossing, segment, etc. the set of allowed ρ is a Zariski open dense set in the representation variety. Looking at the whole diagram means taking a finite intersection of these sets, so it's still Zariski open and dense. This says that it's large enough to meet all conjugacy classes. □

We have now described a way to represent $\mathrm{SL}_2(\mathbb{C})$ representations of tangle complements in terms of diagrams. It's a bit more complicated than the usual way (Wirtinger generators) but there are two good reasons:

1. The shapes are closely related to the shapes of the octahedral decomposition. This gives a geometric interpretation of them, and allows us to use the techniques of [KKY18] to solve for shapings.
2. The shapes can be interpreted as central characters of quantum $\mathfrak{sl}_2$ at a root of unity. This is they key ingredient to constructing quantum holonomy invariants.

2. The octahedral decomposition

Given a tangle T with diagram D the octahedral decomposition gives an ideal triangulation of the complement of T. We refer to [KKY18] for more details on the octahedral decomposition. We now describe how a shaping of a diagram D relates to the octahedral decomposition associated to D.

Remark 2.1. One might worry about the difference between links and tangles. In general a tangle complement is a manifold with boundary, so our triangulations will have "'exposed" faces making up this boundary. Geometrically we should think of this as a totally geodesic boundary, *not* the neighborhood of a cusp torus. When we glue tangles together to get a link these boundaries will all be glued, so we get an ideal triangulation in the usual sense.

Definition 2.2. Let D be a diagram of a tangle T. The *octahedral decomposition* of D is the ideal triangulation of the complement of T minus two points (which we call P_+ and P_-) given by putting an ideal octahedron at every crossing, as in Figure 10.

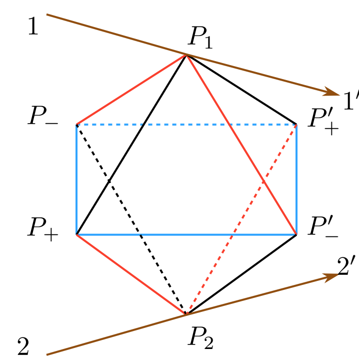

Figure 10: Red segment and blue region edges of an octahedron at a positive crossing. The black edges P_1P_+ and $P_1P'_+$ are glued in the twisted octahedron, and similarly for P_2P_- and $P_2P'_-$.

To make this an ideal triangulation we subdivide each octahedron into tetrahedra. There are two ways of doing this, called the four-term and five-term decompositions. In either case, we can describe the hyperbolic structure on each ideal tetrahedron by assigning it a *shape parameter* $z \in \mathbb{C} \cup \{\infty\}$ (when $z = 0, 1, \infty$ the tetrahedron is geometrically degenerate). This shape parameter is really associated to a pair of opposite edge of the tetrahedron; the other two edge pairs are assigned $1/(1-z)$ and $1 - 1/z$, depending on the combinatorics of the triangulation.

The tetrahedra glue together to give a consistent hyperbolic structure on our tangle complement when the product of shape parameters around each glued edge is 1. The collection of these (one for each edge) are the *gluing equations* of the triangulation. There are also *hyperbolicity equations*[4] at each cusp (for us, link component) that specify the behavior when traveling around the cusp (which we think of as a point at infinity).

We want to relate the gluing equations of the octahedral decomposition of a diagram D. To do this, we assign shapes to the tetrahedra related to the shapes at the crossing. In the four-term

[4] Strictly speaking, the hyperbolicity equation says that the representation is boundary-parabolic, that is has eigenvalues ± 1, because this condition is necessary for a cusped manifold to have a complete finite-volume hyperbolic structure. We consider a more flexible notion, where we might instead request that the representation has eigenvalue $m \in \mathbb{C} \setminus \{0\}$ for m not necessarily 1. In our normalization these were called m^2-hyperbolicity equations in [KKY18].

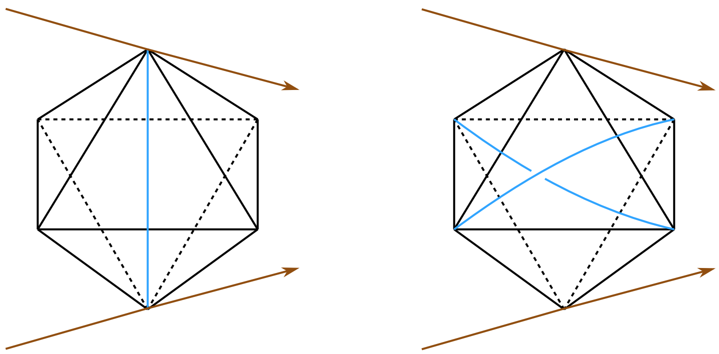

Figure 11: The four-term and five-term decompositions of an octahedron.

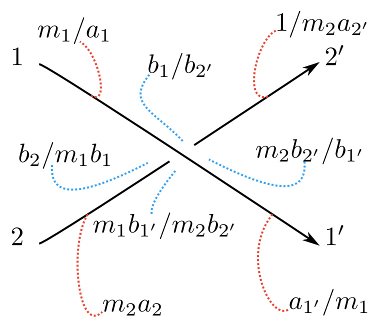

Figure 12: Shapes of edges at a positive crossing. There are four region edges at the four corners and four segment edges below and above the four segments.

triangulation the shapes are given in terms of ratios of the b-variables (and the m-variables); we should think about them as related to the segment variables of [KKY18]. In the five-term triangulation the shapes are instead given in terms of the a-variables (and the m-variables); we should think about these as related to the region variables of [KKY18].

In either case, properties of the braiding map B tell us that the gluing equations of the internal edges of any octahedron are automatically satisfied. In addition, if we take the product of all the shapes at the external edges of the octahedron the values do not depend on whether we used the four or five term decomposition. Explicitly, they are:

Definition 2.3. Let D be a shaped diagram of a link L. Consider a twisted octahedron O at a positive crossing of D. In terms of the characters χ_i of the segments of the crossing, we assign the following shape parameters to the segment and region edges of O:

$$o_1 = \frac{m_1}{a_1} \qquad o_2 = m_2 a_2 \qquad o_{1'} = \frac{a_{1'}}{m_1} \qquad o_{2'} = \frac{1}{m_2 a_{2'}} \tag{6}$$

$$o_N = \frac{b_1}{b_{2'}} \qquad o_W = \frac{b_2}{m_1 b_1} \qquad o_S = \frac{m_1 b_{1'}}{m_2 b_2} \qquad o_E = \frac{m_2 b_{2'}}{b_{1'}} \tag{7}$$

Here by o_j we mean the shape of the segment edge immediately below or above segment j, and by o_k we mean the shape of the region edge near region k. It may be more convenient to consult Figure 12.

At a negative crossing, we instead assign

$$o_1 = m_1 a_1 \qquad o_2 = \frac{m_2}{a_2} \qquad o_{1'} = \frac{1}{m_1 a_{1'}} \qquad o_{2'} = \frac{a_{2'}}{m_2} \tag{8}$$

$$o_N = \frac{b_1}{b_{2'}} \qquad o_W = \frac{b_2}{m_1 b_1} \qquad o_S = \frac{m_1 b_{1'}}{m_2 b_2} \qquad o_E = \frac{m_2 b_{2'}}{b_{1'}} \tag{9}$$

Note that the negative region shapes (9) are the same as in the positive case (7).

This means we can think about the gluing equations of the octahedral decomposition solely in terms of the shapes χ. Our claim is that these are the *same* as finding a shaping of D in the sense of Definition 1.10. Here is a precise statement:

THEOREM 2.4. Let χ be a shaping of D. Then:

- If the shaping is not pinched at any crossing, then it determines a geometrically nondegenerate shaping of the four-term octahedral decomposition.
- If the shaping is not degenerate at any crossing, then it determines a geometrically nondegenerate shaping of the five-term octahedral decomposition. ♢

By *nondegenerate* we mean that no tetrahedron has shape parameter 0, 1, or ∞. We say that a crossing c with incident shapes $\chi_1, \chi_2, \chi_{1'}, \chi_{2'}$ is

- *pinched* if if any of the equations

$$b_2 = m_1 b_1, \quad m_2 b_2 = m_1 b_{1'}, \quad b_{2'} = b_1, \quad m_2 b_{2'} = b_{1'}$$

  hold, in which case all of them do, and

- *degenerate* if either of the equations

$$a_1 = a_{1'} \text{ or } a_2 = a_{2'}$$

  hold, in which case both do.

A degenerate crossing is necessarily pinched but a pinched crossing can be non-degenerate.

REMARK 2.5. While we cannot assign nontrivial shapes to the tetrahedra at a pinched crossing, the shaping and its holonomy representation is still well-defined. This is one advantage to using shapes: we can represent *every* holonomy representation (up to conjugacy).

Proof. The region edges glue together in the regions of the diagram, so the product of all the shape parameters of the region edges around a region needs to be 1. Because they are ratios of b-variables in the product everything cancels. (There are also some factors of m that cancel as well.)

The segment edges glue in a more complicated way involving over and under bridges. However, when we combine their gluing equations with a check that the meridian holonomy is m there is one equation to check at each segment. These equations hold because every segment is assigned a consistent a-variable. □

This theorem tells us how to recover solutions to the gluing equations from a shaping. However, what if we have solutions to the gluing equations in the sense of [KKY18] and want to recover a shaping? It turns out this can always be done:

In the four-term decomposition we think of fixing segment variables z_j for each segment of our diagram, then solving a gluing equation at each segment. The segment variables z_j are up to some factors of m_k exactly the b-variables. The b-variables at a crossing determine the a-variables (we can write the equations for the braiding in such a way that the a-variables are functions of the b-variables) and we can think of the gluing equation at a segment as saying that the a-variables for each half agree. For details, see [McP22, Section 3.3].

In the five-term decomposition we think of fixing region variables w_j for each region equation, then solving a gluing equation around each region. The a-variable of a segment is the ratio of the adjacent region variables (again up to some factors of m_k.) The a-variables at a crossing determine ratios of b-variables, and we can think of the gluing equation of a region as saying that the product of these ratios is 1, so that it is possible to assign a well-defined b-variable to each segment.

Computing shapings from solutions to the gluing equations has a couple of consequences:

1. Because the gluing equations have fewer variables they are usually easier to solve. In [McP22, Section 5] we show how to adapt the gluing equation solutions of [KKY18, Section 6] to compute shapings of twist regions of diagrams.

2. The five-term gluing equations have good nondegeneracy properties, which we can use to prove the existence of geometrically nondegenerate shapings.

Theorem 2.6. A representation $\rho : \pi(L) \to \mathrm{SL}_2(\mathbb{C})$ is *meridian-nontrivial* if $\rho(x) \neq \pm 1$ for any meridian x of L. If $\rho : \pi(L) \to \mathrm{SL}_2(\mathbb{C})$ is merdian-nontrivial and D is *any* diagram of L, then ρ is conjugate to a representation ρ' coming from a shaping of D in which no crossing is degenerate. ♢

Proof. This is exactly the main result of [Yoo21] expressed in our terminology. □

[Yoo21] S. Yoon, "On the potential functions for a link diagram". arXiv DOI

Finally, we should make sure that the geometric holonomy representation associated to the shaping of the octahedral decomposition matches the one we gave in the first section.

Theorem 2.7. The holonomy representation determined by our shape assignment agrees with the representation of Definition 1.8. ♢

Proof. Work in the four-term decomposition. Assign the vertices of each ideal tetrahedron to points in $\mathbb{C} \cup \{\infty\}$ so that their cross-ratios match the assigned shape parameters. Then when gluing two faces f and f' together each has ideal vertices v_0, v_1, v_2 and v'_0, v'_1, v'_2 (indexed so v_j is glued to v'_j), which will not be assigned the same points of $\mathbb{C} \cup \{\infty\}$. There is a unique $g \in \mathrm{PSL}_2(\mathbb{C})$ sending v_j to v'_j for $j = 0, 1, 2$, which we call the *face map*. We can think of $\pi(T)$ as being generated by the face-paring maps of the octahedral decomposition.[5] At a crossing, certain face pairing maps correspond to our paths $x_j^{\pm}$ above and below the segments. It is possible (see [McP22, Theorem 3.6]) to assign the vertices in such a way that the face maps match the matrices $\rho(x_j^{\pm})$ as elements of $\mathrm{PSL}_2(\mathbb{C})$. □

[5] Actually, in this context it is more natural to work with a fundamental groupoid instead of fundamental group: in this version there's one basepoint for each tetrahedron. This is one reason why grouopoids showed up before.

Let's summarize. Suppose we have a tangle T with a representation $\rho : \pi(T) \to \mathrm{SL}_2(\mathbb{C})$ which we have represented in terms of a shaping χ of a diagram D of T.

- The shapes of the tetrahedra in the octahedral decomposition are given explicitly in terms of the parameters of the χ_j.

- As long as ρ is meridian-nontrivial, we can find a *nondegenerate* shaping (one where all tetrahedra are geometrically nondegenerate).

This is useful, but it still does not really explain why to consider our shapings instead of just using the formalism of [KKY18]. To motivate this, we need to first learn some quantum topology.

3. Tangle diagrams and linear algebra

We explain how to think about tangle diagrams as being linear maps. This leads to the Reshetikhin-Turaev [RT90] construction of link invariants like the colored Jones polynomials.

[RT90] N. Y. Reshetikhin and V. G. Turaev, "Ribbon graphs and their invariants derived from quantum groups". DOI

Definition 3.1. In a category we can compose morphisms: if

$$f : V \to W, g : W \to X, \text{ then } g \circ f : V \to X$$

and there are identity morphisms $\mathrm{id}_V : V \to V$ for every object V. Composition is associative.

In a *monoidal* category there is another composition $\otimes$. For any two objects V, W, we have a new object $V \otimes W$, and for any two morphisms $f : V_1 \to W_1$ and $g : V_2 \to W_2$, we get a morphism

$$f \otimes g : V_1 \otimes V_2 \to W_1 \otimes W_2$$

between tensor products. There is also a *unit* object $\mathbf{1}$ with $V \otimes \mathbf{1} = \mathbf{1} \otimes V = V$ for every object V. We also require $\otimes$ to be associative.[6]

[6] Strictly speaking, in general we have $V \otimes \mathbf{1} \cong V$ *isomorphic* to V, not just equal. The isomorphisms implementing this are required to satisfy coherence axioms. Similarly there are are associator *isomorphisms* $V \otimes (W \otimes X) \cong (V \otimes W) \otimes X$. The case where the isomorphisms are equalities is a *strict* monoidal category. These technical issues can be ignored for our purposes.

Example 3.2. Let k be a field and Vect_k the category of finite-dimensional k-vector spaces and linear maps between them. Then defining $\otimes$ to be the usual tensor product of vector spaces makes Vect_k a monoidal category. The tensor unit is the 1-dimensional vector space k.

Example 3.3. We can make tangles in a monoidal category Tang. The objects are nonnegative integers, and a morphism $n \to m$ is a tangle with n boundary points on the left and m on the right. For example, the tangle in fig. 1 is a morphism $3 \to 1$. A link can be viewed as a morphism $0 \to 0$, because there are no endpoints. Composition is given by gluing together at endpoints, as in fig. 2.

The tensor product is given by vertical stacking (in our conventions), as in fig. 2 We have $n \otimes m = n + m$ on objects.

Example 3.4. We can easily extend the previous example to oriented tangles: now boundary points have signs $\pm$ determining whether they are incoming our outgoing, and we can only compose tangles with matching boundary points. Instead of thinking of objects as integers, we think of them as tuples $(\epsilon_1, \epsilon_2, \dots, \epsilon_n)$ of signs $\epsilon_i \in \{1, -1\}$.

Example 3.5. Similarly, we can make shaped tangles into a category: objects are now tuples $((\chi_1, \epsilon_1), (\chi_2, \epsilon_2), \dots)$ of shapes and signs, tangles are required to satisfy the shape equations at crossings, and we can only compose tangles with matching shapes.

Example 3.6. Here's a concrete example. The tangle in fig. 13 is a morphism

$$((\chi_1, +), (\psi_1, +)) \to ((\chi_{N+1}, +), (\psi_{N+1}, +))$$

in the category of shaped tangles. In general, $\chi_{N+1} \neq \chi_1$ and $\psi_{N+1} \neq \psi_1$. In the special case that they *are* equal, we can take the braid closure to get a shaped diagram of the $(2, N)$ torus link (which is a knot if N is odd). More generally, we can solve for χ_{N+1}, ψ_{N+1} as a function of χ_1, ψ_1, and N using W-Fibonacci sequences [McP22, Section 5] by adapting [KKY18, Section 6].

Here "taking the braid closure" can be expressed in the language of monoidal categories. The tangle (in fact, link) in fig. 14 can be thought of as the composition

$$d \circ (t_N \otimes \mathrm{id}) \circ b$$

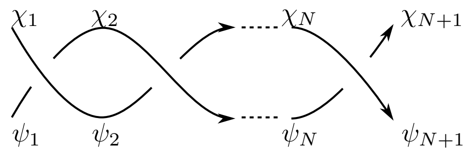

Figure 13: A parallel twist region with N positive crossings. Here we have labeled the shapes of the top and bottom segments as χ_i and ψ_i, respectively.

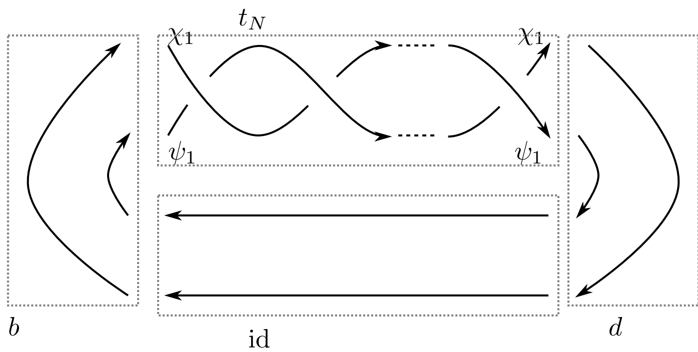

Figure 14: The braid closure can be thought of as composing several tangles together using both types of composition. The bottom two strands can be thought of as an identify morphism.

where $\mathrm{id} = \mathrm{id}_{\psi_1,-} \otimes \mathrm{id}_{\chi_1,-}$ is the appropriately-oriented identity morphism on a pair of strands with shapes χ_1, ψ_1. b is a map

$$b : \emptyset \to ((\chi_1, +), (\psi_1, +), (\psi_1, -), (\chi_1, -))$$

and similarly

$$d : ((\chi_1, +), (\psi_1, +), (\psi_1, -), (\chi_1, -)) \to \emptyset$$

We see that expressing these morphisms in traditional algebraic notation is somewhat difficult: it is much better to use tangle diagrams! (These and related diagrams are often called *string diagrams* in category theory.)

We will return to the category of shaped tangles in the next section; first I want to discuss a relationship between Vect_k and oriented tangles. Pick your favorite object V of Vect_k (i.e. pick some finite-dimensional k-vector space V). We can now interpret tangles as maps between V, the dual space V^*, and their tensor powers. It is enough to understand what to do with the generators in fig. 15, because everything can be built out of them using $\otimes$ and $\circ$.

We send $+$ to the identity map $V \to V$. We think of orientations as corresponding to duals, so $-$ goes to the identity map $V^* \to V^*$. The evaluation and coevaluation maps are more interesting. For example, $\mathrm{ev}^\uparrow$ ought to be a map $V^* \otimes V \to k$; the monoidal unit in Vect_k is k, so the monoidal unit $\emptyset$ of no boundary points ought to go to k. The obvious choice is for $\mathrm{ev}^\uparrow$ to be the evaluation map

$$\mathrm{ev}^\uparrow(f \otimes v) = f(v)$$

and similarly

$$\mathrm{ev}^\downarrow(v \otimes f) = f(v).$$

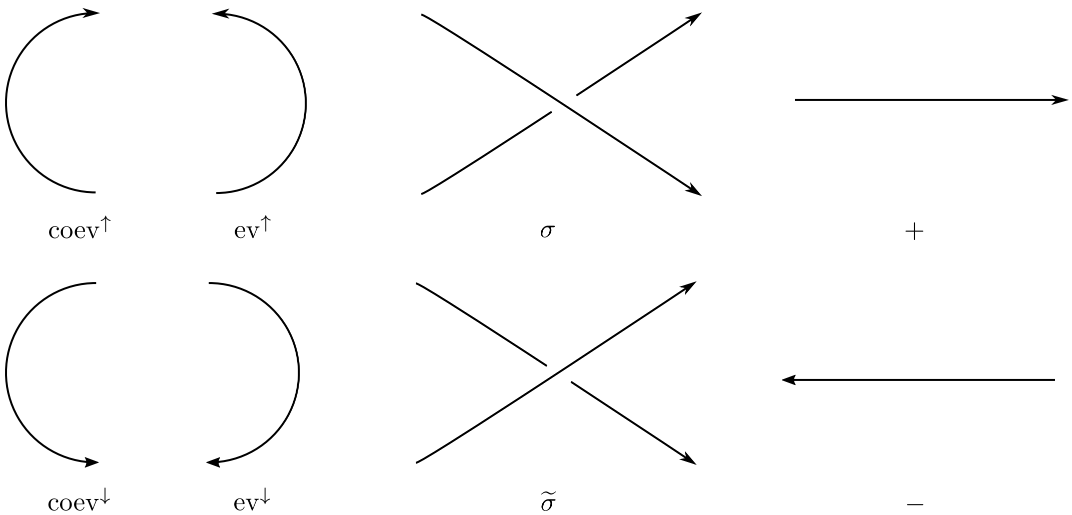

Figure 15: Generators of oriented tangle diagrams.

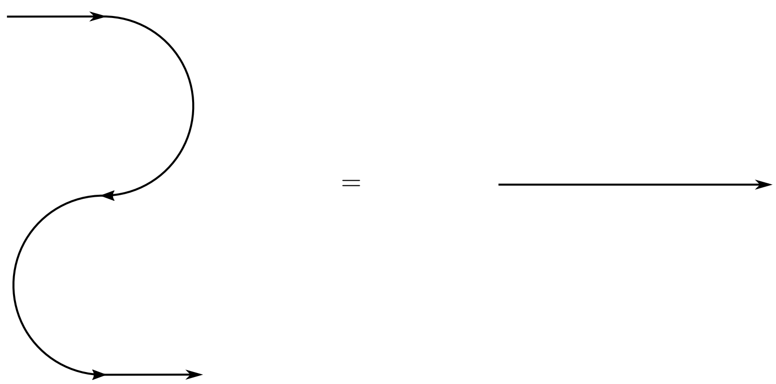

Figure 16: Topological motivation for the relation $(\mathrm{ev}^{\downarrow} \otimes \mathrm{id})(\mathrm{id} \otimes \mathrm{coev}^{\downarrow}) = \mathrm{id}$.

Dual to this, $\mathrm{coev}^{\uparrow}$ is a map $k \to V \otimes V^*$. Pick a basis v_i of V and let v^i be the dual basis. Then we set

$$\mathrm{coev}^{\uparrow}(1) = \sum_i v_i \otimes v^i$$

$$\mathrm{coev}^{\downarrow}(1) = \sum_i v^i \otimes v_i$$

It is not hard to show that this does not depend on the choice of basis. We should do some checking to make sure our assignments match the topology of tangles. For example, from fig. 16 we expect

$$(\mathrm{ev}^{\downarrow} \otimes \mathrm{id})(\mathrm{id} \otimes \mathrm{coev}^{\downarrow}) = \mathrm{id}\,.$$

Again, it it much more natural to describe this graphically! We can check the by hand in this case:

$$\begin{aligned} v_i &\mapsto \sum_j v_i \otimes v^j \otimes v_j \\ &\mapsto \sum_j v^j(v_i) v_j \\ &\mapsto \sum_j \delta_{ij} v_j \\ &= v_i \end{aligned}$$

There are other (similar) axioms to check; the structure we are considering is called a *pivotal* category.[7]

Now that we understand the cap and cup morphisms, it remains to understand the braiding σ. Whatever we assign it should obey the braid relation:

$$(\sigma \otimes \mathrm{id})(\mathrm{id} \otimes \sigma)(\sigma \otimes \mathrm{id}) = (\mathrm{id} \otimes \sigma)(\sigma \otimes \mathrm{id})(\mathrm{id} \otimes \sigma)$$

For Vect_k the obvious choice is to set

$$\sigma(v_1 \otimes v_2) = v_2 \otimes v_1$$

and $\tilde{\sigma} = \sigma^{-1} = \sigma$. We can show that after doing this all the Reidemeister moves are satisfied, so we can consistently assign a tangle

$$T : (\epsilon_1, \ldots, \epsilon_n) \to (\epsilon'_1, \ldots \epsilon'_m)$$

a linear map

$$\mathcal{F}(T) : V^{\epsilon_1} \otimes \cdots \otimes V^{\epsilon_n} \to V^{\epsilon'_1} \otimes \cdots \otimes V^{\epsilon'_m}$$

where we mean $V^1 = V$ and $V^{-1} = V^*$. The assignment $\mathcal{F}$ respects tangle composition and tensor product: it is a *monoidal functor*. Furthermore, if $T = L$ is a *link*, that is a morhpism $\emptyset \to \emptyset$, we have

$$\mathcal{F}(L) : k \to k$$

Identifying linear maps $k \to k$ with their value at $1 \in k$ we can think of this as a scalar. Because $\mathcal{F}$ is compatible with Reidemeister moves it is an invariant of the link L.

The functor $\mathcal{F}$ we gave is a simple example of the *Reshetikhin-Turaev construction* [RT90]. In fact, it is too simple to get interesting link invariants: because $\sigma^2 = \mathrm{id}$, positive and negative crossings are the same! We want to look for a vector space V that admits a more interesting braiding. To do this we need to pass from vector spaces to modules over a more interesting algebra.

[7] There are many types of category with duals; pivotal is sufficient for our purposes.

4. Quantum invariants

To find more interesting braiding maps we consider modules over a certain algebra related to $\mathrm{SL}_2(\mathbb{C})$.

Definition 4.1. Let q be an invertible formal variable. $\mathcal{U}_q(\mathfrak{sl}_2)$ is the algebra over $\mathbb{C}[q, q^{-1}]$ with generators $K^{\pm 1}, E, F$ and relations[8]

$$KK^{-1} = 1, \quad KE = q^2 EK, \quad KF = q^{-2} FK, \quad EF - FE = (q - q^{-1})(K - K^{-1}).$$

[8] The normalization of $EF - FE$ is slightly nonstandard but is more natural for us.

We should think of $\mathcal{U}_q(\mathfrak{sl}_2)$ as a q-deformation of the universal enveloping algebra $\mathcal{U}(\mathfrak{sl}_2)$, which has

$$EH - HE = 2E, \quad FH - HF = -2F, \quad EF - FE = H$$

In the quantum context $K = q^H$. The representation theory of $\mathcal{U}_q(\mathfrak{sl}_2)$ is very similar to that of ordinary $\mathfrak{sl}_2$. In particular, for each integer N we have a highest-weight module V_N of dimension N, generated by a vector v_{N-1} with

$$K \cdot v_{N-1} = q^{N-1}, \quad Ev_{N-1} = 0.$$

Now consider the category $\mathcal{C}$ of finite-dimensional $\mathcal{U}_q(\mathfrak{sl}_2)$-modules, which contains all the modules V_N. (In fact, these are essentially *all* the simple modules in $\mathcal{C}$.) It is a monoidal category with duals, but this is not completely obvious: we need some extra structure.

In general, for A a k-algebra and two A-modules V and W, the tensor product $V \otimes_k W$ is an $A \otimes_k A$-module, but not necessarily an A-module:

$$(a_1 \otimes a_2) \cdot (v \otimes w) = (a_1 \cdot v) \otimes (a_2 \cdot w) \text{ but } a \cdot (v \otimes w) = ?$$

To make $V \otimes W$ a A-module we need an algebra homomorphism $\Delta : A \to A \otimes A$, because then we can set

$$a \cdot (v \otimes w) := \Delta(a) \cdot (v \otimes w).$$

This homomorphism is called a *coproduct* and shows us how to define a product $\otimes$ on the category of A-modules. Similarly, to get a tensor unit $\mathbf{1}$ we make k into a A-module via a homomorphism $\epsilon : A \to k$ called the *counit*:

$$a \cdot x = \epsilon(a)x \text{ for } a \in A, x \in k.$$

This is required to satisfy some relations with Δ to make sure k really is a unit for $\otimes$.[9] For example, to have $V \otimes \mathbf{1} \cong V$ we need

$$a \cdot (v \otimes x) = \Delta(a) \cdot (v \otimes x) = (\mathrm{id} \otimes \epsilon)(\Delta)(a) \cdot v.$$

[9] In fact, you can derive these by drawing the diagrams for multiplication and the unit and reversing the arrows, hence the name.

For duals, the needed structure is a *antipode* S, an anti-homomorphism $S : A \to A$. An algebra with (suitably compatible) coproduct, counit, and antipode is called a *Hopf algebra*. The category of modules over a Hopf algebra becomes a monoidal category with duals.

Example 4.2. Let G be a group. Then the group ring $k[G]$ over k becomes a Hopf algebra via

$$\Delta(g) = g \otimes g, \quad \epsilon(g) = 1, \quad S(g) = g^{-1}.$$

This gives the usual rules for group representations: for tensor products,

$$g \cdot (v \otimes w) = (g \cdot v) \otimes (g \cdot w),$$

the trivial representation is

$$g \cdot x = x,$$

and for duals, if $f \in V^*$ and $v \in V$, then

$$(g \cdot f)(v) := f(g^{-1} \cdot v).$$

Example 4.3. Let $\mathfrak{g}$ be a Lie algebra. Then the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ becomes a Hopf algebra via

$$\Delta(X) = X \otimes 1 + 1 \otimes X, \quad \epsilon(X) = 0, \quad S(X) = -X.$$

PROPOSITION 4.4. $\mathcal{U}_q(\mathfrak{sl}_2)$ is a Hopf algebra with coproduct

$$\Delta(K) = K \otimes K, \quad \Delta(E) = E \otimes K + 1 \otimes E, \quad \Delta(F) = F \otimes 1 + K^{-1} \otimes F,$$

counit

$$\epsilon(K) = 1, \quad \epsilon(E) = \epsilon(F) = 0,$$

and antipode

$$S(E) = -EK^{-1}, \quad S(F) = -KF, \quad S(K) = K^{-1}. \qquad \Diamond$$

Write $\tau(x \otimes y) = y \otimes x$ for the flip map on tensor products. Observe that the coproduct for $k[G]$ and $\mathcal{U}(\mathfrak{g})$ is *cocommutative*: in both cases $\tau\Delta = \Delta$. This means that τ gives a braiding on their module categories; this was the braiding that we earlier discussed, which does not give interesting link invariants.

However, $\mathcal{U}_q(\mathfrak{sl}_2)$ is *not* cocommutative, which suggests we could find a more interesting family of isomorphisms

$$\sigma : V \otimes W \to W \otimes V$$

perhaps ones giving nontrivial braid group representations. Again, this comes from some extra structure on $\mathcal{U}_q(\mathfrak{sl}_2)$.

DEFINITION 4.5. The Hopf algebra $\mathcal{U}_q(\mathfrak{sl}_2)$ has a *universal R-matrix*

$$\mathbf{R} = q^{H \otimes H/2} \sum_{n=0}^{\infty} \frac{q^{n(n-1)/2}}{\{n\}!} (E \otimes F)^n \in \mathcal{U}_\hbar(\mathfrak{sl}_2) \otimes \mathcal{U}_\hbar(\mathfrak{sl}_2) \qquad (10)$$

where $\{n\} := q^n - q^{-n}$ is a quantum integer and $\{n\}! := \{n\}\{n-1\}\cdots\{1\}$ is a quantum factorial.

The key properties of $\mathbf{R}$ are that it intertwines the coproduct and opposite coproduct

$$\mathbf{R}\Delta = \Delta^{\mathrm{op}}\mathbf{R}$$

and satisfies the *Yang-Baxter relation*

$$\mathbf{R}_{12}\mathbf{R}_{13}\mathbf{R}_{23} = \mathbf{R}_{23}\mathbf{R}_{13}\mathbf{R}_{12}$$

which is a version of the braid relation. (Here $\mathbf{R}_{12} = \mathbf{R} \otimes 1$ and so on.)

Strictly speaking $\mathbf{R}$ is not an element of $\mathcal{U}_q(\mathfrak{sl}_2)^{\otimes 2}$ because it contains an infinite sum; we will see some problems later related to this. However, it can be shown that E and F always act nilpotently on any object of $\mathcal{C}$, so the series converges. In particular, for any objects V, W of $\mathcal{C}$ we get a map

$$R_{V,W} : V \otimes W \to V \otimes W, \quad v \otimes w \mapsto \mathbf{R} \cdot (v \otimes w).$$

If we set $c_{V,W} = \tau R_{V,W}$, then $c_{V,V}$ is a $\mathcal{U}_q(\mathfrak{sl}_2)$-module isomorphism $V \otimes V \to V \otimes V$ satisfying the braid relation

$$(c_{V,V} \otimes \mathrm{id})(\mathrm{id} \otimes c_{V,V})(c_{V,V} \otimes \mathrm{id}) = (\mathrm{id} \otimes c_{V,V})(c_{V,V} \otimes \mathrm{id})(\mathrm{id} \otimes c_{V,V}).$$

The map c is the braiding used to define quantum link invariants. To define them, pick an object $V \in \mathcal{C}$ (that is, pick a representation of $\mathcal{U}_q(\mathfrak{sl}_2)$). Then

- strands are assigned the identity map $V \to V$ or $V^* \to V^*$ depending on orientation,
- cups and caps are assigned evaluation and coevaluation maps,

- crossings between V and W are assigned $c_{V,V}$ or $c_{V,V}^{-1}$ depending on sign. Depending on the orientations we might also consider maps c_{V,V^*}.

Together these give a functor $\mathcal{F}_V : \mathrm{Tang} \to \mathcal{C}$. In particular, when L is a link, we get a linear map $\mathcal{F}_V(L) : \mathbb{C}[q, q^{-1}] \to \mathbb{C}[q, q^{-1}]$ which we identify with the scalar $\mathcal{F}_V(L)(1)$. This scalar is an invariant of L.

Example 4.6. Let V_2 be the irreducible 2-dimensional representation of $\mathcal{U}_q(\mathfrak{sl}_2)$, which is analogous to the defining 2-dimensional representation of $\mathfrak{sl}_2$. Then the invariant $\mathcal{F}_{V_2}(L)$ is the *Jones polynomial* of L. By checking relations between the linear maps $c_{V,V}$, $c_{V,V}^{-1}$, and $\mathrm{id}_{V\otimes V}$ we can recover Jones' skein relation.

Concretely, one can compute $\mathcal{F}_{V_2}(L)$ by:

1. expressing L as the closure of a braid β (say on n strands),
2. using the functor $\mathcal{F}_{V_2}$ to get a map
$$\mathcal{F}_{V_2}(\beta) : V_2^{\otimes n} \to V_2^{\otimes n},$$
3. then use $\mathrm{coev}_V^{\uparrow}$ and $\mathrm{ev}_V^{\downarrow}$ to close up the braid, which we think of as taking a *quantum trace* of the linear map $\mathcal{F}_{V_2}(\beta)$.

Part of the data of a functor $\mathcal{F} : \mathrm{Tang} \to \mathcal{C}$ from tangles to another category is a braid group representation, which we use here.

Remark 4.7. There are some subtleties here that did not show up in our vector space example. One is that $\mathrm{coev}^{\uparrow}$ and $\mathrm{coev}^{\downarrow}$ and $\mathrm{ev}^{\uparrow}$ and $\mathrm{ev}^{\downarrow}$ can differ; this is related to the fact that in Vect_k there is a standard choice of identification $V^{**} \cong V$ but in general such a choice is part of the data of a pivotal category.

A related but more complicated issue is that our invariant will depend on the framing of the link (i.e. it will change under Reidemeister I moves). In practice, the framing dependence can be removed by changing the normalization of $c_{V,V}$; in general, the structure allowing one to do this makes our category a *ribbon category*. We refer to [GPV13, Section 1] for more on pivotal categories and [Oht01, Chapter 4] for more on ribbon categories.

[GPV13] N. Geer, B. Patureau-Mirand, and A. Virelizier, "Traces on ideals in pivotal categories". DOI

[Oht01] T. Ohtsuki, *Quantum Invariants*. DOI

Example 4.8. In parallel to the unique N-dimensional irreducible representation of $\mathfrak{sl}_2$ for each nonnegative integer N there is a N-dimensional irrep of $\mathcal{U}_q(\mathfrak{sl}_2)$ we write V_N. The associated quantum invariant $\mathcal{F}_{V_N}(L)$ is the *Nth colored Jones polynomial*.

Remark 4.9. There are many normalization conventions for the colored Jones polynomials; some are unimportant (like q versus $q^{1/2}$) but others are more significant. For example, in our formalism it is most natural to set

$$\mathcal{F}_{V_N}(\bigcirc) = \mathrm{ev}_V^{\downarrow} \circ \mathrm{coev}_V^{\uparrow} = \frac{q^N - q^{-N}}{q - q^{-1}}$$

where $\bigcirc$ is the unknot. However, in the version appearing in the volume conjecture we need $\mathcal{F}_{V_N}(\bigcirc) = 1$.

The value

$$\mathrm{ev}_V^{\downarrow} \circ \mathrm{coev}_V^{\uparrow}$$

is called the *quantum dimension* of V; to see why, consider the corresponding quantity in Vect_k: for v_i a basis and v^i the dual basis,

$$\mathrm{ev}^{\downarrow} \circ \mathrm{coev}^{\uparrow}(1) = \mathrm{ev}^{\downarrow}\left(\sum_i v_i \otimes v^i\right) = \sum_i v^i(v_i) = \dim V.$$

Here is a famous application. Let K be a hyperbolic knot in S^3 and set $\xi = \exp(\pi i/N)$ a $2N$th root of unity. Write

$$J_N(K) := \left.\frac{\mathcal{F}_{V_N}(K)}{\mathcal{F}_{V_N}(\bigcirc)}\right|_{q=\xi} \in \mathbb{C}$$

for the colored Jones polynomial of K, normalized so the value of the unknot is 1, then evaluated at $q = \xi$. If we did not do this we would get a trivial invariant, because $\mathcal{F}_{V_N}(\bigcirc)|_{q=\xi} = 0$.

Conjecture 4.10 (Volume conjecture).

$$\lim_{N\to\infty} \log \frac{|J_N(K)|}{N} = \frac{\operatorname{Vol}(S^3 \setminus K)}{2\pi}. \qquad \lozenge$$

One motivation for quantum holonomy invariants is to study this conjecture.

5. Quantum holonomy invariants

We now explain how to modify the Reshetikhin-Turaev construction to use shaped tangles, leading to the invariants of [Bla+20; McP21]. When we computed the Nth colored Jones polynomial we picked a single representation $V = V_N$ for every segment in our diagram. However, in a shaped tangle, segments are now decorated by shapes $\chi = (a, b, m)$. To generalize, we want a *family* of representations V_χ indexed by shapes. Such a family arises from the representation theory of $\mathcal{U}_\xi(\mathfrak{sl}_2)$ when $q = \xi = \exp(\pi i/N)$ is a root of unity.

[McP21] C. McPhail-Snyder, "$\mathrm{SL}_2(\mathbb{C})$-holonomy invariants of links". arXiv

In general, if A is a finite-dimensional algebra over $\mathbb{C}$ (or any algebraically closed field) then a good way to classify its representations is to study its center $Z(A)$. Let V be a simple A-module (i.e. an irreducible representation of A). By Schur's Lemma, any $z \in Z(A)$ acts by a *scalar* $\chi(z)$ on V, so we get a central character

$$\chi_V : Z(A) \to \mathbb{C}$$

determined by V. Studying these characters is frequently useful.

Example 5.1. For q not a root of unity, the center $Z(\mathcal{U}_q(\mathfrak{sl}_2))$ of quantum $\mathfrak{sl}_2$ is generated by the *Casimir*

$$\Omega = EF + q^{-1}K + qK^{-1} = FE + qK + q^{-1}K^{-1}.$$

On the irreducible N-dimensional representation V_N the element Ω acts by $q^N + q^{-N}$; the only[10] modules with this value are V_N and V_N^*, so by classifying the action of the center we have gone a long way towards classifying all modules.

[10] Actually, there are 2 more given by changing some signs.

At a root of unity, the center gets much larger.

Proposition 5.2. When $q = \xi = \exp(\pi i/N)$, the subalgebra

$$\mathcal{Z}_0 := \mathbb{C}[K^N, K^{-N}, E^N, F^N] \subset \mathcal{U}_\xi(\mathfrak{sl}_2)$$

is central. The algebra $\mathcal{Z}_0$ is large in the sense that $\mathcal{U}_\xi(\mathfrak{sl}_2)/\ker\chi$ has dimension N^2 for any χ, and the whole center

$$Z(\mathcal{U}_\xi(\mathfrak{sl}_2)) = \mathcal{Z}_0[\Omega]/(\text{polynomial relation})$$

is generated by $\mathcal{Z}_0$ and the Casimir element Ω, modulo a degree N polynomial relation given by a Chebyshev polynomial. $\lozenge$

For V a simple $\mathcal{U}_\xi(\mathfrak{sl}_2)$-module, we get a central character

$$\chi : \mathcal{Z}_0 \to \mathbb{C}$$

determined by the values

$$\chi(K^N) = \kappa, \quad \chi(E^N) = \epsilon, \quad \chi(K^N F^N) = \phi.$$

We think of these values as corresponding to the holonomy matrices

$$\begin{bmatrix} \kappa & 0 \\ \phi & 1 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & \epsilon \\ 0 & \kappa \end{bmatrix}$$

hence to the shape (a, b, m) with $a = \kappa$ and b, m satisfying

$$\begin{aligned} \epsilon &= (a - m)b \\ \phi &= (a - 1/m)/b \end{aligned}$$

We will see shortly where the shape coordinates a, b, m come from; for now let's keep working with κ, ϵ, ϕ. Consider the group

$$\mathrm{SL}_2(\mathbb{C})^* := \left\{ \left(\begin{bmatrix} \kappa & 0 \\ \phi & 1 \end{bmatrix}, \begin{bmatrix} 1 & \epsilon \\ 0 & \kappa \end{bmatrix} \right) \middle| \kappa \neq 0 \right\} \subset \mathrm{GL}_2(\mathbb{C}) \times \mathrm{GL}_2(\mathbb{C})$$

It is closely related to our factorization of $\pi(D)$ into $\Pi(D)$; we think of the first factor as corresponding to x^+ and the second to x^-. We can identify characters $\chi : \mathcal{Z}_0 \to \mathbb{C}$ with points of $\mathrm{SL}_2(\mathbb{C})^*$. This identification is a group homomorphism: the coproduct gives a product on characters

$$(\chi_1 \cdot \chi_2)(z) := (\chi_1 \otimes \chi_2)(\Delta(z))$$

which agrees with the product of $\mathrm{SL}_2(\mathbb{C})^*$.

Theorem 5.3. Let $\chi : \mathcal{Z}_0 \to \mathrm{SL}_2(\mathbb{C})$ be a character. Write

$$\chi^+ = \begin{bmatrix} \chi(K^N) & 0 \\ \chi(K^N F^N) & 1 \end{bmatrix}, \quad \chi^- = \begin{bmatrix} 1 & \chi(E^N) \\ 0 & \chi(K^N) \end{bmatrix}$$

and $\operatorname{tr} \chi = \operatorname{tr} \chi^+ (\chi^-)^{-1}$. Let V be any simple $\mathcal{U}_\xi(\mathfrak{sl}_2)$-module with character χ. Then:

1. Ω acts on V by some $\omega \in \mathbb{C}$ with

   $$\mathrm{Cb}_N(\omega) = \operatorname{tr} \chi$$

   We should think of this as a Nth root of the eigenvalues of the meridian holonomy $\chi^+(\chi^-)^{-1}$, because the Chebyshev polynomial is characterized by

   $$\mathrm{Cb}_N(\mu + \mu^{-1}) = \mu^N + \mu^{-N}.$$

2. If $\operatorname{tr} \chi \neq \pm 2$, there are N simple $\mathcal{U}_\xi(\mathfrak{sl}_2)$-modules with character χ. They are classified by the N roots ω of the equation $\mathrm{Cb}_N(\omega) = \operatorname{tr} \chi$. We denote them by $V(\chi, \omega)$.
3. In the boundary-parabolic case $\operatorname{tr} \chi = \pm 2$ there are many more modules, including a family of N-dimensional representations on which Ω acts by $\xi^{N-1} + \xi^{-(N-1)}$. ◊

Concretely, $V(\chi, \omega)$ is generated by a "highest-weight" vector v_0; a basis is given by $\{v_0, \ldots v_{N-1}\}$ with

$$K \cdot v_k = \kappa^{1/N} \xi^{2k} v_k$$

However, we no longer have $E \cdot v_0 = 0$. Instead $E \cdot v_0 = \epsilon v_{N-1}$. This is quite different from ordinary $\mathfrak{sl}_2$ representation theory.

We now see how to get a quantum holonomy invariant. Because shapes χ determine $\mathcal{Z}_0$-characters, given a *shaped* tangle diagram, we can assign each strand with shape χ the module $V(\chi, \omega)$. To determine ω we need some extra data; geometrically, this is a choice μ of Nth root of the meridian eigenvalue m. (We set $\omega = \mu + \mu^{-1}$.) This leads towards the following theorem:

THEOREM 5.4. Let L be a link and $\rho : \pi(L) \to \mathrm{SL}_2(\mathbb{C})$ a representations. Recall our choice of integer N related to the order of ξ. For each component L_i of L with meridian eigenvalue m_i, choose a Nth root μ_i of m_i. In the boundary-parabolic case $m_i = \pm 1$ there are some restrictions on the choice of μ_i. Then there is a *quantum holonomy invariant*

$$\mathcal{F}(L, \rho, \{\mu_i\}_i) \in \mathbb{C}$$

defined up to Nth roots of unity. ◊

EXAMPLE 5.5. If ρ is the trivial representation we recover the colored Jones polynomial at a root of unity, as in the volume conjecture.

EXAMPLE 5.6. When $N = 2$, we recover the Reidemeister torsion, in the sense that

$$\mathcal{F}(L, \rho, \{\mu_i\}_i)\mathcal{F}(\overline{L}, \overline{\rho}, \{\mu_i\}_i) = \tau(S^3 \setminus L, \rho)$$

where $(\overline{L}, \overline{\rho})$ is the mirror image of (L, ρ) and τ is the Reidemeister torsion of the link complement twisted by ρ. This is the main result of [McP20].

[McP20] C. McPhail-Snyder, "Holonomy invariants of links and nonabelian Reidemeister torsion". arXiv

Some technical issues:

1. The braiding is quite difficult to define; we will discuss this next.
2. The quantum dimension of the modules $V(\chi, \omega)$ vanishes, so our invariants do as well if we apply the usual RT construction. To fix this, use *modified dimensions*.
3. "Nice" categories of representations like those of a group or simple Lie algebra in dimension 0 usually have a property called *semisimplicity*, which means all decomposable objects split as direct sums. The category of $\mathcal{U}_\xi(\mathfrak{sl}_2)$-modules is *not* semisimple, which is related to the vanishing quantum dimensions. This lack of semisimplicity causes significant technical issues as well. We mention this because the quantum holonomy invariants are sometimes called "non-semisimple quantum invariants."
4. The invariant is only defined up to phase ambiguities; these have to do with the braiding.

Because of these issues, in particular (1), very few examples have been computed. One reason for my interest in octahedral decompositions is to be able to compute more.

6. THE QUANTUM HOLONOMY BRAIDING

The key step in defining our quantum invariants is understanding the braidings c. In this case, we have a *family* of braidings

$$c_{\chi_1, \chi_2} : V(\chi_1, \omega_1) \otimes V(\chi_2, \omega_2) \to V(\chi_{2'}, \omega_2) \otimes V(\chi_{1'}, \omega_1)$$

parametrized by characters. Unlike in the usual case, these are *not* computed by the action of the universal R-matrix $\mathbf{R}$. The reason is that E and F no longer act nilpotently, so the power series does not converge! There is still a way to use it to define a braiding, but it is more complicated than before.

Lemma 6.1. The map $\mathcal{R} : \mathcal{U}_q(\mathfrak{sl}_2)^{\otimes 2} \to \mathcal{U}_q(\mathfrak{sl}_2)^{\otimes 2}$ given by conjugation

$$\mathcal{R}(x) = \mathbf{R} x \mathbf{R}^{-1}$$

by the universal R-matrix gives an algebra automorphism that still makes sense for $q = \xi$. In particular, $\mathcal{R}$ acts nontrivially on $\mathcal{Z}_0 \otimes \mathcal{Z}_0$. It induces a map $B : (\chi_1, \chi_2) \mapsto (\chi_{2'}, \chi_{1'})$ via

$$(\chi_{1'} \otimes \chi_{2'})\mathcal{R} = (\chi_1 \otimes \chi_2). \qquad \lozenge$$

When the domain is appropriately restricted the map B is *exactly* the braiding map on shapes. We will explain this more shortly.

First, we see how to recover the braiding $c_{\chi_1,\chi_2} = \tau R_{\chi_1,\chi_2}$ on modules. (Here as before τ is the flip $\tau(v \otimes w) = w \otimes v$.) We call R_{χ_1,χ_2} a *holonomy R-matrix* because it depends on the holonomy representation of the tangle complement. It is characterized by the intertwining property

$$\mathcal{R}(x \cdot v) = \mathcal{R}(x) \cdot R_{\chi_1,\chi_2}(v) \text{ for every } x \in \mathcal{U}_\xi(\mathfrak{sl}_2)^{\otimes 2}, v \in V_{\chi_1} \otimes V_{\chi_2}.$$

In fact, this is enough to determine c_{χ_1,χ_2} up to an overall scalar. Unfortunately, in order to compute the quantum invariant we need to

1. determine an explicit formula for c_{χ_1,χ_2}, and
2. use this (or some other method) to fix the normalization.

This project is ongoing; a partial solution is given in [McP21, Chapter 3]. The key idea (joint with N. Reshetikhin) is to use a certain presentation of $\mathcal{U}_\xi(\mathfrak{sl}_2)$ in terms of q-Weyl algebras. This presentation is very closely related (via cluster algebras) to the octahedral decomposition. In particular, the holonomy braiding c_{χ_1,χ_2} factors into a product of four *quantum dilogarithms*, which are matrix functions analogous to the dilogarithm used in the computation of complex volume. Understanding this computation requires understanding the octahedral decomposition, which was my original motivation for studying it. In addition, the presence of quantum dilogarithms suggests a relationship to the *quantum hyperbolic invariants* of Baseilhac and Benedetti [BB04; BB05; BB11].

[McP21] C. McPhail-Snyder, “$\mathrm{SL}_2(\mathbb{C})$-holonomy invariants of links”. arXiv

[BB04] S. Baseilhac and R. Benedetti, “Quantum hyperbolic invariants of 3-manifolds with $PSL(2, C)$-characters”. DOI

[BB05] S. Baseilhac and R. Benedetti, “Classical and quantum dilogarithmic invariants of flat $\mathrm{PSL}(2, \mathbb{C})$-bundles over 3-manifolds”. arXiv DOI

[BB11] S. Baseilhac and R. Benedetti, “The Kashaev and quantum hyperbolic link invariants”. arXiv

This computation is the source of the shape coordinates.

Definition 6.2. The *extended Weyl algebra* is the algebra $\mathcal{W}_q$ generated over $\mathbb{C}[q, q^{-1}]$ by a central invertible element z and invertible x, y subject to the relation

$$xy = q^2 yx.$$

Proposition 6.3. The map $\phi : \mathcal{W}_q \to \mathcal{U}_q(\mathfrak{sl}_2)$ given by

$$K \mapsto x \qquad E \mapsto qy(z - x) \qquad F \mapsto y^{-1}(1 - z^{-1}x^{-1})$$

is an algebra homomorphism. It acts on the Casimir by

$$\Omega \mapsto qz + (qz)^{-1}.$$

At a $2N$th root of unity $q = \xi$ the center of $\mathcal{W}_\xi$ is generated by x^N, y^N, and z. The automorphism ϕ takes the center of $\mathcal{U}_\xi(\mathfrak{sl}_2)$ to the center of $\mathcal{W}_\xi$. Explicitly,

$$\begin{aligned} \phi(K^N) &= x^N \\ \phi(E^N) &= y^N(x^N - z^N) \\ \phi(F^N) &= y^{-N}(1 - z^{-N}x^{-N}). \end{aligned} \qquad \lozenge$$

In particular, a central character $\chi : \mathcal{W}_\xi \to \mathbb{C}$ with

$$\chi(x^N) = a, \quad \chi(y^N) = b, \quad \chi(z^N) = m$$

gives a $Z(\mathcal{U}_\xi(\mathfrak{sl}_2))$-character via

$$\begin{aligned} \chi(K^N) &= \kappa = a \\ \chi(E^N) &= \epsilon = (a - m)b \\ \chi(F^N) &= \phi = (a - 1/m)/b \end{aligned}$$

and this identifies shapes with $\mathcal{Z}_0$-characters.[11] We can derive the formula for the braiding map B by pulling back the automorphism $\mathcal{R}$ to shapes via ϕ; details are given in [McP22, Section 4].

[11] Similarly the value $\chi(z) = \mu$ determines $\chi(\Omega) = \mu + \mu^{-1}$.

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