What is Lie algebra cohomology and why should you care?

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Two important results in the basic structure theory of Lie algebras are the existence of Levi decompositions and complete reducibility for modules of semisimple Lie algebras. While they have elementary proofs, both ultimately use ideas from homological algebra. This document attempts to explain that connection to someone who knows about Lie algebras but not much about homological algebra.

None of the ideas in this text are mine, but I first learned about this approach in the Math 261B (Lie Groups and Lie Algebras) course taught by Mark Haiman in Spring 2016 at UC Berkeley. He described it as the "Reduced Shakespeare Company" approach to homological algebra.

Throughout we assume that all Lie algebras are finite-dimensional and defined over a field k of characterisitc zero. All modules are finite-dimensional.

1 Complete reducibility

One of the reasons the representation theory of semisimple Lie algebras is nice is that every representation decomposes into a direct sum of irreducibles. This is *not* true for solvable algebras, as one can see from examples like

$$\mathfrak{g} = \left\{ \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} : a, b \in k \right\} \subseteq \mathfrak{gl}(k^2).$$

$\mathfrak{g}$ has a natural representation on the space $W = k^2$ of column vectors by matrix multiplication, but the invariant submodule V spanned by $\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]$ doesn't have a complement: there is no submodule U of W with $W \cong V \oplus U$.

Another way to look at this is to examine the short exact sequence

$$0 \to V \to W \xrightarrow{\pi} V' \to 0,$$

where $V' = W/V$. This sequence does not *split*: there is no morphism $\epsilon : V' \to W$ such that $\pi\epsilon = \mathrm{id}_{V'}$. In the notation of the previous paragraph, we would have $\operatorname{im} \epsilon = U$.

We can therefore state complete reducibility as follows:

Theorem 1.1 (Complete reducibility)**.** *Let $\mathfrak{g}$ be a semisimple Lie algebra and let E_1, E_2 be $\mathfrak{g}$-modules. Then any exact sequence*

$$0 \to E_1 \to E \xrightarrow{\pi} E_2 \to 0$$

splits.

Such a sequence is called an *extension of E_1 by E_2*. The simplest possible case is a split extension. For modules this corresponds to a direct sum, while for groups and algebras it's a semidirect product.

Reduction to a simple case. We want to reduce to the case where E_2 is the trivial module k. To do this, consider the subspace W of $\mathrm{Hom}_k(E, E_1)$ of operators that act by scalars on E_1, and the subspace $V \subseteq W \subseteq \mathrm{Hom}_k(E, E_1)$ of operators that act by zero. We then get an exact sequence

$$0 \to V \to W \to k \to 0,$$

unless $E_1 = 0$, in which case we're done anyway. If this sequence splits, then we have an element $\phi \in W$ which is $\mathfrak{g}$-invariant and maps to 1 in k, i.e. a $\mathfrak{g}$-module homomorphism $E \to E_1$ whose restriction to E_1 is the identity. The kernel of ϕ is the desired complimentary submodule to E_1, so our original sequence splits. □

Proving complete reducibility therefore amounts to proving it in the case where one module is trivial, so let's examine the case

$$0 \to V \to W \to k \to 0$$

in detail. While the above may not split as a sequence of *modules*, it will split as a sequence of vector spaces, so can write everything in W as a pair (v, λ) with $v \in V$ and $\lambda \in k$. Then the action of $\mathfrak{g} \in X$ is given by

$$X \cdot (v, \lambda) = (X \cdot v + \lambda f(X) + X \cdot \lambda, 0) = (Xv + \lambda f(X), 0)$$

where f is a linear map $\mathfrak{g} \to V$ that describes the failure of the subspace k to be a submodule.

We can say more about the map f. In order for the previous formula to give a representation of Lie algebras, we need

$$XY(v, \lambda) - YX(v, \lambda) = [X, Y](v, \lambda).$$

That is,

$$\begin{aligned} XY(v, \lambda) - YX(v, \lambda) &= (XYv + \lambda X f(Y) - YXv + \lambda Y f(X), 0) \\ &= ([X, Y] + \lambda \left(X f(Y) - Y f(X) \right), 0), \end{aligned}$$

needs to agree with

$$[X, Y](v, \lambda) = ([X, Y]v + \lambda f([X, Y]), 0).$$

Therefore $f : \mathfrak{g} \to V$ is a valid structure map for the extension when it satisfies the *1-cocycle condition*

$$Xf(Y) - Yf(X) = f([X, Y]), \tag{1}$$

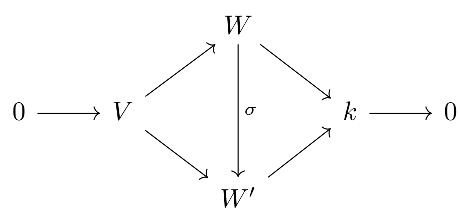
in which case we call f a *1-cocyle*. We write $Z^1(\mathfrak{g}, V)$ for the space of 1-cocyles. Every 1-cocyle describes an extension of V by k, but not uniquely.

For example, particulary simple structure maps f correspond to split exact sequences. If $f(X) = Xu$ for some fixed $u \in V$, then

$$X(v, \lambda) = (Xv + \lambda Xu, 0)$$

so changing the inclusion map $k \hookrightarrow W$ from $\lambda \mapsto (0, \lambda)$ to $\lambda \mapsto (-\lambda u, \lambda)$ will give a splitting of $\mathfrak{g}$-modules.

Slightly more formally, we say that two extensions W, W' are equivalent if there is a morphism $\sigma : W \to W'$ making the diagram



commute; if it does then σ is necessarily an isomorphism.

In order to make explicit calculations, choose preferred splittings $W = V \oplus k$ and $W' = V \oplus k$ of vector spaces, so that the $\mathfrak{g}$-module structures on W and W' are given by 1-cocycles $f, f' : \mathfrak{g} \to V$ respectively. We can now describe σ as a block matrix $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$. Then if the left-hand triangle commutes we must have $A = \mathrm{id}_V$ and $C = 0$, while the right-hand triangle gives $D = 1$. The remaining block B is a linear map $k \to V$, that is a fixed element $u \in V$.

Now in order for σ to be a morphism of $\mathfrak{g}$-modules, we need

$$\sigma(X \cdot (v, \lambda)) = \sigma(Xv + \lambda f(X), 0) = (Xv + \lambda f(X), 0)$$

to agree with

$$X \cdot \sigma(v, \lambda) = X \cdot (v + \lambda u, \lambda) = (Xv + \lambda Xu + \lambda f'(X), 0)$$

where the $\mathfrak{g}$-action in the second equation is that of W'. This occurs exactly when $f(X) - f'(X) = u$ for every $X \in \mathfrak{g}$.

We therefore call maps $f : \mathfrak{g} \to V$ of the form $f(X) = Xu$ for $u \in W$ *1-coboundaries*, and write the space of them as $B^1(\mathfrak{g}, V)$. (Of course, $B^1(\mathfrak{g}, V)$ is the same thing as V, but viewed as a set of maps $\mathfrak{g} \to V$ using the $\mathfrak{g}$-action.)

The previous discussion shows that the failure of the sequence $0 \to V \to W \to k \to 0$ to split is measured by the vector space

$$H^1(\mathfrak{g}, V) = \frac{Z^1(\mathfrak{g}, V)}{B^1(\mathfrak{g}, V)}$$

of 1-cocyles modulo 1-coboundaries. The notation suggests that there are other, higher cohomology groups and some kind of general way of constructing them, which we will get to later on. For now, observe that the proof of Theorem 1.1 reduces to showing that $H^1(\mathfrak{g}, V)$ vanishes for semisimple $\mathfrak{g}$.

2 Levi decomposition

It is possible to broadly divide Lie algebras into two types: *solvable* and *semisimple.* In general a Lie algebra is neither solvable nor semisimple, but has parts of each type. The *radical* $\operatorname{Rad}(\mathfrak{g})$ of $\mathfrak{g}$ is the maximal solvable ideal, so we get a short exact sequence

$$0 \to \operatorname{Rad}(\mathfrak{g}) \to \mathfrak{g} \to \mathfrak{g}/\operatorname{Rad}(\mathfrak{g}) \to 0$$

with $\mathfrak{s} = \mathfrak{g}/\operatorname{Rad}(\mathfrak{g})$ semisimple. If we know that this sequence splits, then we can to some degree deal with the solvable and semisimple parts of $\mathfrak{g}$ separately. Such a splitting is called a Levi decomposition, and it always exists:

Theorem 2.1 (Levi decomposition)**.** *For any Lie algebra* $\mathfrak{g}$ *the sequence*

$$0 \to \operatorname{Rad}(\mathfrak{g}) \to \mathfrak{g} \to \mathfrak{g}/\operatorname{Rad}(\mathfrak{g}) \to 0$$

of Lie alebras splits. In particular, this means that $\mathfrak{g}$ *is isomorphic to a semidirect product* $\operatorname{Rad}(\mathfrak{g}) \rtimes (\mathfrak{g}/\operatorname{Rad}(\mathfrak{g}))$ *with the second factor semisimple.*

Reduction to a simple case. As before, we reduce to a simple case, then explain how that case is determined by cohomology. Write $\mathfrak{r} = \operatorname{Rad}(\mathfrak{g})$ and $\mathfrak{s} = \mathfrak{g}/\mathfrak{r}$. Let $\mathfrak{r}' = [\mathfrak{r}, \mathfrak{r}]$ be the derived ideal of $\mathfrak{r}$; because $\mathfrak{r}$ is solvable it is a proper ideal. Taking the quotient by $\mathfrak{r}'$ gives a short exact sequence

$$0 \to \mathfrak{a} \to \mathfrak{g}/\mathfrak{r}' \to \mathfrak{s} \to 0$$

since $\mathfrak{s}/\mathfrak{r}' = \mathfrak{s}$. Notice that $\mathfrak{a} = \mathfrak{r}/\mathfrak{r}'$ is abelian, and that any splitting of the quotient sequence lifts to a spliting of the original sequence. It follows that we can reduce to the case where $\operatorname{Rad}(\mathfrak{g})$ is abelian. □

Notice that in the above we can think of $\mathfrak{a}$ as a $\mathfrak{s}$-module, via the action induced by the adjoint action of $\mathfrak{g}$, and since $\mathfrak{a}$ is abelian it has essentially no bracket. Therefore, we change notation slightly to consider short exact sequences of the form

$$0 \to \mathfrak{a} \to \mathfrak{e} \to \mathfrak{g} \to 0$$

where $\mathfrak{g}$ is a semisimple Lie algebra and $\mathfrak{a}$ is a $\mathfrak{g}$-module, thought of as an abelian Lie algebra. If we can show that every such sequence splits, we have proved Theorem 2.1.

As before, we can choose a vector space spliting $\mathfrak{e} = \mathfrak{a} \oplus \mathfrak{g}$. Then since $\mathfrak{a}$ is abelian, the Lie algebra structure of $\mathfrak{e}$ will be given by a map $f : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{a}$:

$$[(u, X), (v, Y)] = (f(X, Y) + Xv - Yu, [X, Y]).$$

We use lower-case letters for elements of $\mathfrak{a}$ because we are thinking of it as a $\mathfrak{g}$-module.

For the above to give a Lie bracket we at least need $f(X,Y) = -f(Y,X)$, so that f is a k-linear map $\bigwedge^2\mathfrak{g} \to \mathfrak{a}$. As

$$\begin{aligned}[[(u,X),(v,Y)],(w,Z)] &= [(f(X,Y)+Xv-Yu,[X,Y]),(w,Z)] \\ &= ([X,Y]w - Zf(X,Y) - ZXv + ZYu + f([X,Y],Z),[[X,Y],Z]),\end{aligned}$$

the Jacobi identity requires

$$f([X,Y],Z)+f([Y,Z],X)+f([Z,X],Y)-Zf(X,Y)-Yf(Z,X)-Xf(Y,Z) = 0. \tag{2}$$

These conditions are sufficient for $\mathfrak{a} \oplus \mathfrak{g}$ to be a Lie algebra, so we call a k-linear map $f : \bigwedge^2\mathfrak{g} \to \mathfrak{a}$ satisfying (2) a *2-cocycle* and write $Z^2(\mathfrak{g},\mathfrak{a})$ for the space of 2-cocycles.

Just as before, any equivalence of extensions σ will be given by some k-linear map $g : \mathfrak{g} \to \mathfrak{a}$:

$$\sigma(v,X) = (v + g(X), X),$$

and two extensions with structure 2-cocycles f, f' will be equivalent when

$$\sigma(f(X,Y)+Xv-Yu,[X,Y]) = (f(X,Y)+Xv-Yu+g([X,Y]),[X,Y])$$

is equal to

$$\begin{aligned}[\sigma(u,X),\sigma(v,Y)] &= [(v+g(X),X),(u+g(Y),Y)] \\ &= (f'(X,Y)+Xu-Yv+Xg(Y)-Yg(X),[X,Y]),\end{aligned}$$

i.e. when f and f' differ by a map $(\delta g)(X,Y) = g([X,Y]) - Xg(Y) + Yg(X)$, where $g : \mathfrak{g} \to \mathfrak{a}$. We therefore define the *coboundary operator* $\delta : \mathrm{Hom}_k(\mathfrak{g},\mathfrak{a}) \to \mathrm{Hom}_k(\bigwedge^2\mathfrak{g},\mathfrak{a})$ by

$$(\delta g)(X,Y) = g([X,Y]) - Xg(Y) + Yg(X), \tag{3}$$

and call the image of $\mathrm{Hom}_k(\mathfrak{g},\mathfrak{a})$ under δ the space $B^2(\mathfrak{g},\mathfrak{a})$ of *2-coboundaries*.

This discussion shows that proving Theorem 2.1 reduces to showing that the space

$$H^2(\mathfrak{g},\mathfrak{a}) = \frac{Z^2(\mathfrak{g},\mathfrak{a})}{B^2(\mathfrak{g},\mathfrak{a})}$$

vanishes for semisimple $\mathfrak{g}$. Our notation is not totally arbitrary: we previously had an implicit 1-coboundary operator $\delta : \mathfrak{a} = \mathrm{Hom}_k(k,\mathfrak{a}) \to \mathrm{Hom}_k(\mathfrak{g},\mathfrak{a})$ given by

$$\delta(u)(X) = Xu,$$

and

$$\begin{aligned}\delta(\delta u)(X,Y) &= (\delta u)([X,Y] - X(\delta u)(Y) + Y(\delta u)(X) \\ &= [X,Y]u - XYu + YXu = 0,\end{aligned}$$

so $\delta^2 = 0$, as expected for a coboundary operator.

3 Constrution of cohomology and the connection with Ext

Now that we've seen the motivating examples, let's construct the cohomology groups in general. The *Chevalley-Eilenberg complex* for the Lie algebra $\mathfrak{g}$ with coefficients in the module V is the cochain complex

$$0 \to V \to \mathrm{Hom}_k(\mathfrak{g}, V) \to \mathrm{Hom}_k(\textstyle\bigwedge^2 \mathfrak{g}, V) \to \cdots \to \mathrm{Hom}_k(\textstyle\bigwedge^k \mathfrak{g}, V) \to \cdots$$

with the coboundary operators defined by the formula

$$\begin{aligned}\delta^n f(X_0, \ldots, X_n) = &\sum_{k=0}^{n} (-1)^k X_k f(X_0, \ldots, \widehat{X_k}, \ldots, X_n) \\ &+ \sum_{k,l=0}^{n} (-1)^{k+l} f([X_k, X_l], X_0, \ldots, \widehat{X_k}, \ldots, \widehat{X_l}, \ldots, X_n)\end{aligned}$$

It is a tedious but elementary exercise to check that $\delta^n \delta^{n-1} = 0$, so we can define the *cohomology groups of* $\mathfrak{g}$ *with values in* V by

$$H^n(\mathfrak{g}, V) = \frac{\ker \delta^n}{\operatorname{im} \delta^{n-1}}.$$

These of course agree with our previous definitions. Notice also that

$$H^n(\mathfrak{g}, V_1 \oplus V_2) = H^n(\mathfrak{g}, V_1) \oplus H^n(\mathfrak{g}, V_2)$$

since in a direct sum of $\mathfrak{g}$-modules both terms of the differential decompose. This result will be of limited usefulness until we have proved complete reducibility, however.

At this point, we can prove both of our theorems by showing that $H^n(\mathfrak{g}, V)$ is zero for all V. Unfortunately, this statement is false! In general the groups $H^n(\mathfrak{g}, k)$ can be nonzero. For the case of $k = \mathbb{R}$, it can be shown that $H^n(\mathfrak{g}, \mathbb{R})$ is isomorphic to the de Rahm cohomology $H^n_{\mathrm{dR}}(G, \mathbb{R})$, where G is the compact, connected, simply connected Lie group with Lie algebra $\mathfrak{g}$.

(The important step in proving this is to prove that the usual de Rahm cohomology of G is isomorphic to the G-invariant cohomology, by way of averaging with an invariant volume form. Once this is done, you can check that the coboundary operators above agree with the exterior derivative for forms evaluated on invariant vector fields.)

However, $H^n(\mathfrak{g}, V)$ will always vanish when $\mathfrak{g}$ is semisimple and V has no trivial submodules, and we can show that $H^n(\mathfrak{g}, V) = 0$ for every V if $n = 1, 2$. Unfortunately, it is difficult to show that the cohomology vanishes directly from our definition.We can at least see the following:

Proposition 3.1. *If* $\mathfrak{g}$ *is semisimple,* $H^1(\mathfrak{g}, k) = 0$.

Proof. Suppose $f : \mathfrak{g} \to k$ is a cochain. In order for it to be a cocycle we need

$$Xf(Y) - Yf(X) - f([X,Y]) = -f([X,Y]) = 0$$

for every $X, Y \in \mathfrak{g}$. But by semisimplicity $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$, so f must be the zero map. Since there are no nontrivial cocycles $H^1(\mathfrak{g}, k) = 0$.[1] □

At this point we need to introduce some abstract homological algebra. Specifically, we will show that our cohomology groups $H^n(\mathfrak{g}, V)$ are the same as the Ext groups $\mathrm{Ext}^n_{\mathcal{U}\mathfrak{g}}(k, V)$ and then explain why the Ext groups vanish. For the reader who has encountered some homological algebra, this might be expected: the usual interest in the Ext groups is that they classify extensions, which is how we motivated the cohomomlogy groups in the first place.

For now we work slightly more generally. Let U be an associative k-algebra and M a (left) U-module. We can abstract the idea of "generators and relations" for M into a *free resolution,* which is an exact sequence

$$\cdots \to F_2 \to F_1 \to F_0 \to M \to 0$$

of U-modules, with the F_k free U-modules, i.e. isomorphic to a direct sum of copies of U.[2] The idea is that F_0 is the span of the generators with no relations, F_1 the set of relations on the generators, F_2 the set of relations on the relations, etc. This is one way to show that all modules have free resolutions, in fact: If we set $F_{-1} = M_0 = M$, $M_{k+1} = \ker(F_k \to F_{k-1})$, and F_k to be the free module on a generating set of M_k, then $F_\bullet$ (the complex indexed by $\bullet$) is a free resolution of M.

Given another U-module N, we can obtain another complex by applying the contravariant functor $\mathrm{Hom}(-, N)$ to a free resolution $F_\bullet$ of M:

$$\mathrm{Hom}_U(F_\bullet, N) = 0 \to \mathrm{Hom}_U(M, N) \to \mathrm{Hom}_U(F_0, N) \to \mathrm{Hom}_U(F_1, N) \to \cdots$$

In general this complex will *not* be exact, and the failure to be exact is measured by the cohomology groups

$$\mathrm{Ext}^k_U(M, N) = H^i(\mathrm{Hom}_U(F_\bullet, N)).$$

In particular, $\mathrm{Ext}^0_U(M, N) = \mathrm{Hom}_U(M, N)$.

Proposition 3.2. $\mathrm{Ext}^k_U(M, N)$ *does not depend on the choice of free resolution and is functorial in M and N.*

Proof. Proving this is somewhat beyond the scope of this document, but I might add an appendix later. The basic idea to to show that any two free resolutions of M have a chain map between them (construct one inductively by freeness) and that this chain map induces an isomorphism on cohomology (while you make aribtrary choices, they all cancel out.) □

[1] This is basically a disguised version of the direct proof: any sequence $0 \to k \to W \to k \to 0$ splits because any vector space splitting works.

[2] Sometimes a free resolution is instead defined as a sequence $\cdots \xrightarrow{\delta_2} F_1 \xrightarrow{\delta_1} F_0 \to 0$ that is exact everywhere except at 0, where $H_0(F_\bullet) = \ker \delta_1 = M$.

We now consider the case where $U = \mathcal{U}\mathfrak{g}$ is the universal enveloping algebra of $\mathfrak{g}$, so that M and N are $\mathcal{U}\mathfrak{g}$-modules, i.e. $\mathfrak{g}$-modules. There are general reasons why Ext classifies extensions, but in our case it's clearer to use a specific free resolution to show that $\mathrm{Ext}^k_{\mathcal{U}\mathfrak{g}}(k, V)$ is equal to the Lie algebra cohomology, which we already know classifies extensions.

Specifically, consider the following free resolution (the *Koszul resolution*) of the trivial representation k:

$$\cdots \to \mathcal{U}\mathfrak{g} \otimes \bigwedge^k \mathfrak{g} \xrightarrow{d_k} \cdots \to \mathcal{U}\mathfrak{g} \otimes \bigwedge^2 \mathfrak{g} \xrightarrow{d_2} \mathcal{U}\mathfrak{g} \otimes \mathfrak{g} \xrightarrow{d_1} \mathcal{U}\mathfrak{g} \xrightarrow{\epsilon} k \to 0$$

where the tensor products are over k and with the differentials given by

$$\begin{aligned} d_k(X_1 \wedge \cdots \wedge X_k) = &\sum_i (-1)^{i-1} X_i \otimes (X_i \wedge \cdots \widehat{X_i} \cdots \wedge X_k) \\ &- \sum_{i<j} (-1)^{i-j+1} 1 \otimes ([X_i, X_j] \wedge X_1 \wedge \cdots \widehat{X_i} \cdots \widehat{X_j} \cdots \wedge X_k) \end{aligned} \tag{4}$$

It's not hard to check that these are well-defined and square to zero; what is harder is checking exactness.

Lemma 3.3. *The complex described by (4) is exact.*

Proof. Suppose we have a complex $F_\bullet(t)$ of k-vector spaces depending on a parameter $t \in k$, and that the vector spaces do not change with t, only the (matrix coefficients of) the differentials d_k. The ranks of the d_k will increase for values of t described by zeros of some polynomials, which means that on a Zariski-open set they are constant. In particular, exactness is a Zariski open condition.

Therefore, consider the related complex

$$\begin{aligned} d_k(X_1 \wedge \cdots \wedge X_k) = &\sum_i (-1)^{i-1} X_i \otimes (X_i \wedge \cdots \widehat{X_i} \cdots \wedge X_k) \\ &- t \sum_{i<j} (-1)^{i-j+1} 1 \otimes ([X_i, X_j] \wedge X_1 \wedge \cdots \widehat{X_i} \cdots \widehat{X_j} \cdots \wedge X_k) \end{aligned}$$

which corresponds to the Lie algebra $\mathfrak{g}_t$ that is the same as $\mathfrak{g}$ but with bracket $[X, Y]_t = t[X, Y]$. For nonzero t, $\mathfrak{g}_t \cong \mathfrak{g}$, while $\mathfrak{g}_0$ is abelian. In particular, at $t = 0$ the complex is obviously exact (all the H^i have dimension zero.) By our general principle it is also exact in a Zariski open neighborhood of zero, which is nonempty because k is an infinite field (we assumed at the beginning that it was characteristic zero.)[3] Thus it's exact for some $\mathfrak{g}_t \cong \mathfrak{g}$. □

Corollary 3.4. $\mathrm{Ext}^n_{\mathcal{U}\mathfrak{g}}(k, V) \cong H^n(k, V)$.

Proof. Applying $\mathrm{Hom}_k(-, V)$ to the Koszul resolution gives the Chevalley-Eilenberg complex. □

[3] Actually this works for all k, because we can replace k by its algebraic closure and that won't affect exactness.

4 Casimir elements and vanishing of cohomology

At this point, we are almost done. We can show easily that $\mathrm{Ext}^n(k, V) = \mathrm{Ext}^n_{\mathcal{U}\mathfrak{g}}(k, V)$ vanishes for nontrivial simple modules V, and we can then use this and some homological algebra to build up the more general vanishing results.

Proposition 4.1. *Let $\mathfrak{g}$ be semisimple and let V be a nontrivial simple $\mathfrak{g}$-module. Then $\mathrm{Ext}^n(k, V)$ vanishes for all n.*

Proof. The idea is to find a *Casimir element*, a central element of $\mathcal{U}\mathfrak{g}$ that acts trivially on k and nontrivially on V. To do this, consider the trace form (closely related to the Killing form)

$$\beta_V(X, Y) = \mathrm{tr}(X|_V Y|_V)$$

defined on $\mathfrak{g}$. This form is symmetric. It won't always be nondegenerate, but we can get close enough: Since $\mathfrak{g}$ is semisimple, we can write it as a direct product $\mathfrak{g} = \mathfrak{g}_1 \times \cdots \times \mathfrak{g}_k$ of simple algebras.[4] Some $\mathfrak{g}_l$ acts nontrivially on V, so by Cartan's second criterion and the simplicity of $\mathfrak{g}_l$ β_V is nondegenerate on $\mathfrak{g}_l$.

Now choose a basis X_i of $\mathfrak{g}_l$ and let X^i be the dual basis with respect to the form β_V. We can now define the Casimir element

$$c = \sum_i X_i \otimes X^i.$$

c is then central by the $\mathfrak{g}$-invariance of β_V, and

$$\mathrm{tr}(c|_V) = \sum_i \mathrm{tr}(X_i|_V X^i|_V) = \sum_i \beta(X_i, X^i) = \dim \mathfrak{g}_l \neq 0,$$

so it's nonzero. It follows by Schur's lemma that c acts on M as an automorphism.

c acts as an automorphism of M, so its eigenvalues (which could lie in the algebraic closure of k) are all nonzero. On the other hand c annihilates k, so its eigenvalues on k are all zero. As a consequence the characteristic polynomials f, g of c on k, V are relatively prime. By functoriality, c also acts on $\mathrm{Ext}^n(k, V)$, so this space must be annihilated by both $f(c)$ and $g(c)$. But f and g are relatively prime, so this forces $\mathrm{Ext}^n(k, V) = 0$. □

Corollary 4.2 (First Whitehead lemma)**.** *If $\mathfrak{g}$ is semisimple, then $H^1(\mathfrak{g}, V) = 0$ for any V.*

We provide both a direct proof, and one using more homological algebra. In both cases, the idea is to show that it follows from $H^1(\mathfrak{g}, V) = 0$ for *simple* V (including $V = k$), which we already know.

[4] This is a reduciblity result, but not one that needs cohomology: given a simple ideal of $\mathfrak{g}$, we can take the complementary ideal to be the orthogonal complement under the Killing form. This *does* rely on k being characteristic zero.

Proof not by homological algebra. Recall that $H^1(\mathfrak{g}, V)$ measures extensions $0 \to V \to W \to k \to 0$. If $U \subset V$ is a proper submodule, then we can use induction on dimension to find a splitting of the sequence

$$0 \to V/U \to W/U \to k \to 0$$

and this splitting will then lift to one of the original sequence. It therfore suffices to show that all sequences $0 \to V \to W \to k \to 0$ with V simple split, the desired reduction. □

Proof by homological algebra. Alternately we can use another property of Ext. It assigns to a short exact sequence $0 \to A \to B \to C \to 0$ a long exact sequence

$$0 \to \mathrm{Ext}^0(C, V) \to \mathrm{Ext}^0(B, V) \to \mathrm{Ext}^0(A, V) \to \mathrm{Ext}^1(C, V) \to \cdots$$

This is a general feature of things that act like cohomology. For example, one can think of a (nice) subspace $A \subseteq X$ of a topological space as being a short exact sequence $A \to X \to X/A$, and there is a corresponding long exact sequence in singular homology and cohomology.

In particular this long exact sequence can be used to induct. If V is simple we are done. If $U \subset V$ is a proper submodule, then there is a short exact sequence $0 \to U \to V \to V/U \to 0$ of modules, the corresponding long exact sequence includes the terms

$$\cdots H^1(\mathfrak{g}, U) \to H^1(\mathfrak{g}, V) \to H^1(\mathfrak{g}, V/U) \cdots$$

and by induction on dimension we can assume $H^1(\mathfrak{g}, U) = H^1(\mathfrak{g}, V/U) = 0$. □

Corollary 4.3 (Second Whitehead lemma)**.** *If $\mathfrak{g}$ is semisimple, then $H^2(\mathfrak{g}, V) = 0$ for any V.*

Proof. Now that we have complete reducibility of $\mathfrak{g}$-modules, we only need to prove the vanishing for simple V, and by Proposition 4.1 the only case left is $V = k$, that is a sequence

$$0 \to k \to \mathfrak{e} \to \mathfrak{g} \to 0.$$

But then k lies in the center of $\mathfrak{e}$, so the adjoint action of $\mathfrak{e}$ factors through $\mathfrak{g} = \mathfrak{e}/k$. Since $\mathfrak{e}$ is completely reducible as a $\mathfrak{g}$-module, the sequence splits and $H^2(\mathfrak{g}, k) = 0$. □

References

[1] William Fulton and Joe Harris, *Representation Theory: A First Course.* Springer-Verlag, 1991

[2] Jean-Pierre Serre, *Lie Algebras and Lie Groups*, second edition. Springer-Verlag, 2006.

[3] Charles A. Weibel, *An Introduction to Homological Algebra.* Cambridge University Press, 1995.