REPRESENTATIONS OF THE GENERAL LINEAR SUPERGROUP

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The subject of this talk is to relate certain *generalized Khovanov diagram algebras* to the representation theory of the general linear supergroup $GL(m|n)$. The talk is based on a series of four papers by Brundan and Stroppel where they do much more than this: [BS11a], [BS09],[BS11b], and [BS10]. In order to motivate this connection, we discuss some basic aspects of representation theory in the 'super' world. However, given the time constraints, we will skip over many topics and points, and few proofs can be discussed. The interested listener can read more about representations of lie superalgebras from Serganova's survey in [CCC+17] (pp. 125-177), or the books [CW12], [Mus12].

All vector spaces and algebras in this talk are over $\mathbb{C}$, the complex numbers (for simplicity). As always, please be careful, with what you read, as there are possibly errors in what is written. See references for further details.

1. The Category $SVect$; Lie Superalgebras

We start by defining the symmetric monoidal category $SVect$. The objects, which we call *super vector spaces*, are $\mathbb{Z}_2$-graded vector spaces $V = V_{\overline{0}} \oplus V_{\overline{1}}$. For a homogeneous element $x \in V$, we write $\overline{x} \in \mathbb{Z}_2$ for its degree. The morphisms are degree-preserving maps of vector spaces. The monoidal structure is given by $\mathbb{Z}_2$-graded tensor product: for V, W super vector spaces, we define the super vector space $V \otimes W$ with grading:

$$(V \otimes W)_{\overline{0}} = V_{\overline{0}} \otimes W_{\overline{0}} \oplus V_{\overline{1}} \otimes W_{\overline{1}}, \quad (V \otimes W)_{\overline{1}} = V_{\overline{1}} \otimes W_{\overline{0}} \oplus V_{\overline{0}} \otimes W_{\overline{1}}$$

Then the object $\mathbb{C} := \mathbb{C}^{1|0} = \mathbb{C} \oplus 0$ is the unit object. So far nothing of interest has happened. To enter the super world, we take the following braiding isomorphism:

$$\sigma_{VW} : V \otimes W \xrightarrow{\sim} W \otimes V, \qquad v \otimes w \mapsto (-1)^{\overline{v}\overline{w}} w \otimes v$$

Since $\sigma_{WV} \circ \sigma_{VW} = \mathrm{id}_{V \otimes W}$, we get an induced action of S_n on $V^{\otimes n}$. Hence any construction we can do classically that can be phrased in terms of representations of S_n, we can do here. In particular, all Schur functors $S_\lambda(V)$ (e.g. symmetric or exterior powers) are defined for a super vector space V (althought they are *not* the same as the classical Schur functors in the category of vector spaces with the usual braiding.)

Remark 1.1. There is a parity shift functor $\Pi : SVect \to SVect$ defined by $(\Pi V)_{\overline{i}} = V_{\overline{i+1}}$, and given $f : V \to W$, $\Pi(f)$ is f once again but on the parity shifted spaces. To understand how this functor respects the symmetric monoidal structure, it is best to define it as $\Pi V = \mathbb{C}^{0|1} \otimes V$. This required a choice, i.e. we could have tensored on the right instead.

In this case, the functor Π^n admits a natural action of S_n. Further, we get natural isomorphisms

$$S_\lambda(\Pi V) \cong \Pi^n S_{\lambda'}(V)$$

for partition λ.

In particular, we may define a lie algebra in $SVect$ to be an object $\mathfrak{g}$ with a map $[-,-] : \Lambda^2\mathfrak{g} \to \mathfrak{g}$ such that $[-,[-,-]] : \mathfrak{g}^{\otimes 3} \to \mathfrak{g}$ is killed by $(1+\sigma+\sigma^2) \in \mathbb{C}[S_3]$, where $\sigma \in S_3$ is a three-cycle (this is the *super* Jacobi identity). We call a lie algebra in the category of $SVect$ a *Lie superalgebra.*

Some observations: if $\mathfrak{g}$ is a Lie superalgebra, then $\mathfrak{g}_{\overline{0}}$ is a Lie algebra, $\mathfrak{g}_{\overline{1}}$ is a $\mathfrak{g}_{\overline{0}}$-module, and the commutator defines a map $[-,-] : S^2\mathfrak{g}_{\overline{1}} \to \mathfrak{g}_{\overline{0}}$ which is $\mathfrak{g}_{\overline{0}}$-equivariant (here $S^2\mathfrak{g}_{\overline{1}}$ is the usual second symmetric power).

An example: given a superalgebra $A = A_{\overline{0}} \oplus A_{\overline{1}}$, we can construct a Lie superalgebra from it via its commutator in $SVect$ (otherwise know as the *supercommutator*), which can be computed to be:

$$[a,b] = ab - (-1)^{\overline{a}\overline{b}}ba$$

Given a super vector space V, we may define $\underline{\mathrm{End}}(V)$ to be all vector space endomorphisms of V (not necessarily of degree 0). Then $\underline{End}(V)$ is a superalgebra, where the degree $\overline{i}$ part consists of maps $V \to V$ of degree $\overline{i}$. The lie superalgebra $\mathfrak{gl}(V)$ is one given by taking $\underline{End}(V)$ with supercommutator. We call $\mathfrak{gl}(V)$ the general linear superalgebra from V. In particular, we define $\mathfrak{gl}(m|n) := \mathfrak{gl}(\mathbb{C}^{m|n})$, where $\mathbb{C}^{m|n} = \mathbb{C}^m \oplus \mathbb{C}^n$.

A representation of a Lie superalgebra $\mathfrak{g}$ is a super vector space V with a Lie superalgebra homomorphism $\mathfrak{g} \to \mathfrak{gl}(V)$. We also may define the enveloping algebra $\mathcal{U}\mathfrak{g}$ as the superalgebra $T^\bullet\mathfrak{g}/\langle [x,y] - (xy - (-1)^{\overline{x}\overline{y}}yx)\rangle$. As usual, representations of $\mathfrak{g}$ are equivalent to modules over its enveloping algebra. Further, there is a PBW theorem for superalgebras, and an important consequence of it is that $\mathcal{U}\mathfrak{g}$ is a free $\mathcal{U}\mathfrak{g}_0$-module of finite rank $2^{\dim \mathfrak{g}_{\overline{1}}}$.

2. Representations of The General Linear Supergroup

We now discuss $\mathfrak{g} = \mathfrak{gl}(m|n)$ further. It is presented by matrices of the form

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$$

where $\mathfrak{g}_{\overline{0}}$ consists of matrices of the form:

$$\left[\begin{array}{c|c} A & 0 \\ \hline 0 & D \end{array}\right]$$

and we observe $\mathfrak{g}_{\overline{0}} \cong \mathfrak{gl}(m) \times \mathfrak{gl}(n)$ as a Lie algebra. Also, $\mathfrak{g}_{\overline{1}}$ consists of matrices of the form:

$$\left[\begin{array}{c|c} 0 & B \\ \hline C & 0 \end{array}\right]$$

which is isomorphic to the $\mathfrak{g}_{\overline{0}}$-module $\mathbb{C}^n \otimes (\mathbb{C}^m)^* \oplus (\mathbb{C}^n)^* \otimes \mathbb{C}^m$.

In fact, $\mathfrak{g}$ has a nice $\mathbb{Z}$-grading: $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$, where $\mathfrak{g}_0 = \mathfrak{g}_{\overline{0}}$, and

$$\mathfrak{g}_1 = \left[\begin{array}{c|c} 0 & B \\ \hline 0 & 0 \end{array}\right], \quad \mathfrak{g}_{-1} = \left[\begin{array}{c|c} 0 & 0 \\ \hline C & 0 \end{array}\right]$$

Goal: We are interested in understanding the category of finite-dimensional $\mathfrak{g}$-modules which are integrable over $\mathfrak{g}_0$. We call this category $Rep(GL(m|n))$. We do not discuss $GL(m|n)$ explicitly, but its finite-dimensional representation theory is equivalent $Rep(GL(m|n))$ as we have defined it (via the theory of super Harish-Chandra pairs).

Some remarks:

(1) $Rep(GL(m|n))$ is *not* a semisimple category. However all objects have finite-length.
(2) $Rep(GL(m|n))$ has a highest weight theory similar to that of semisimple Lie algebras.
(3) In fact, $Rep(GL(m|n))$ is a so-called *highest weight category* in the sense of Cline, Parshall, and Scott (see [CPS88]). This underlies many of the ideas of this talk.
(4) The simple objects of $Rep(GL(m|n))$ (up to parity) are in (reasonably natural) bijection with the simple objects of $Rep(GL(m) \times GL(n))$.
(5) $Rep(GL(m|n))$ has a enough projectives- the functor $\mathrm{Ind}_{\mathfrak{g}_{\overline{0}}}^{g}(-) = \mathcal{U}\mathfrak{g} \otimes_{\mathcal{U}\mathfrak{g}_{\overline{0}}} (-)$ is exact and left adjoint to restriction, and therefore takes $\mathfrak{g}_{\overline{0}}$-modules to projectives.
(6) An object in $Rep(GL(m|n))$ is projective if and only if it is injective. Therefore projectives admit no non-trivial extensions with any object.

Now we setup some notation to get specific. Choose $\mathfrak{h}$ to be the subalgebra of all diagonal matrices, and write $\epsilon_1, \dots, \epsilon_m, \delta_1, \dots, \delta_n$ for the usual basis of $\mathfrak{h}^*$ (here ϵ_i is dual to the ith diagonal entry and δ_i is dual to the $(m+i)$th diagonal entry). Define

$$\rho = -\epsilon_2 - 2\epsilon_3 - \cdots - (m-1)\epsilon_m + (m-1)\delta_1 + \cdots + (m-n)\delta_n$$

(here ρ is (roughly) the half sum of all even roots minus the half sum of all odd roots). We have inner product on $\mathfrak{h}^*$ given by

$$(\epsilon_i, \epsilon_j) = \delta_{ij}, \quad (\epsilon_i, \delta_j) = 0, \quad (\delta_i, \delta_j) = -\delta_{ij}$$

Write $P = \mathbb{Z}\langle \epsilon_1, \dots, \epsilon_m, \delta_1, \dots, \delta_n \rangle$ for the collection of integral weights, P^+ for the integral weights which are dominant over $\mathfrak{g}_0$, which we recall consists of all weights of the form:

$$\lambda = a_1\epsilon_1 + \cdots + a_m\epsilon_m + b_1\delta_1 + \cdots + b_n\delta_n, \quad \text{where } a_1 \geq a_2 \geq \cdots \geq a_m, \;\; b_1 \geq b_2 \geq \cdots \geq b_n$$

equivalently

$$(\lambda+\rho, \epsilon_1) > (\lambda+\rho, \epsilon_2) > \cdots > (\lambda+\rho, \epsilon_m), \quad (\lambda+\rho, \delta_1) < (\lambda+\rho, \delta_2) < \cdots < (\lambda+\rho, \delta_n)$$

Now, let $\lambda \in P^+$ be a dominant integral weight. Then we may construct the finite-dimensional $\mathfrak{g}_{\overline{0}}$-module $L_0(\lambda)$, irreducible of highest weight λ. Extend $L_0(\lambda)$ to a $\mathfrak{g}_0 \oplus \mathfrak{g}_1$-module by letting $\mathfrak{g}_1$ act by 0. Then we define the *Kac*-module as

$$K(\lambda) := \mathrm{Ind}_{\mathfrak{g}_0 \oplus \mathfrak{g}_1}^{\mathfrak{g}} L_0(\lambda) = \mathcal{U}\mathfrak{g} \otimes_{\mathcal{U}(\mathfrak{g}_0 \oplus \mathfrak{g}_1)} L_0(\lambda).$$

By the PBW theorem mentioned above, $K(\lambda)$ is finite-dimensional. It turns out that $K(\lambda)$ is indecomposable and has a unique irreducible finite-dimensional quotient, which we call $L(\lambda)$. Further, every finite-dimensional quotient is isomorphic to $L(\lambda)$ for some $\lambda \in P^+$ (up to parity shift).

Write $P(\lambda)$ for projective cover of $L(\lambda)$- that is, an indecomposable projective such that it's largest semisimple quotient is $L(\lambda)$. We have a sequence of surjections:

$$P(\lambda) \twoheadrightarrow K(\lambda) \twoheadrightarrow L(\lambda)$$

When we have $P(\lambda) = K(\lambda) = L(\lambda)$ we say that λ is a *typical* weight. There is a numerical condition that is equivalent to typicality, to be discussed below.

Our goal is to understand the category $Rep(GL(m|n))$. For this, we look at the algebra

$$K = \mathrm{End}_{\mathfrak{g}}(\bigoplus_{\lambda \in P^+} P(\lambda)) \cong \bigoplus_{\lambda,\mu \in P^+} \mathrm{Hom}_{\mathfrak{g}}(P(\lambda), P(\mu))$$

(where the isomorphism comes from $\mathrm{Hom}_{\mathfrak{g}}(P(\lambda), P(\mu)) = 0$ for all but finitely many μ if we fix λ). Write $Mof(K)$ for the category of finite-dimensional K-modules. Then by some abstract nonsense, we have an equivalence of categories

$$Rep(GL(m|n)) \xrightarrow{\sim} Mof(K)$$

a representation V goes to $\mathrm{Hom}_{\mathfrak{g}}(\bigoplus_{\lambda \in P^+} P(\lambda), V)$.

Therefore, we would like to understand $\mathrm{Hom}_{\mathfrak{g}}(P(\lambda), P(\mu))$ for arbitrary dominant weights λ, μ.

Example 2.1. Here we discuss the case $Rep(GL(1|1))$. In $\mathfrak{gl}(1|1)$ we have the element $h = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ which is central. Hence its action separates representations. One can show that h acts by a non-zero scalar on $L(\lambda)$, then λ is typical, i.e. $L(\lambda)$ is projective, and hence has no non-trivial extensions with other objects.

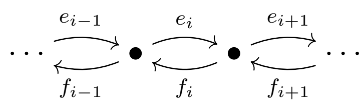
Write $\mathcal{O}_0$ for the subcategory of $Rep(GL(1|1))$ where h acts by 0. This is referred to as the 'principal block' of $Rep(GL(1|1))$. The irreducibles are $L(n) := L(n(\epsilon_1 - \delta_1))$, where $n \in \mathbb{Z}$, along with their parity shifts. These are one-dimensional modules and we concentrate them in degree $n(\text{mod } 2)$. The Kac module $K(n)$ and projective $P(n)$ look like:

$$K(n) = \boxed{\begin{array}{c} L(n) \\ \hline L(n-1) \end{array}} \qquad P(n) = \boxed{\begin{array}{cc} \multicolumn{2}{c}{L(n)} \\ \hline L(n-1) & L(n+1) \\ \hline \multicolumn{2}{c}{L(n)} \end{array}},$$

It follows that

$$\mathrm{End}_{\mathfrak{g}}(P(n)) \cong \mathbb{C}[x]/(x^2) \quad \dim \mathrm{Hom}_{\mathfrak{g}}(P(n), P(n-1)) = \dim \mathrm{Hom}_{\mathfrak{g}}(P(n), P(n+1)) = 1,$$

and $\mathrm{Hom}_{\mathfrak{g}}(P(n), P(m)) = 0$ if $m \neq n-1, n, n+1$. Computing the Ext-quiver, one can show that $Rep(GL(1|1))$ is equivalent to finite-dimensional representations of the infinite quiver



such that $f_i e_i + e_{i-1} f_{i-1} = 0$.

3. Khovanov's Diagram Algebra

In come the diagram algebras. This is a rewrite of the original notes. Since I'm not that good at tex-ing, I won't be drawing any pictures here. Watch the talk for some diagrams! Or look at Brundan-Stroppel.

We fix once and for all a line which can, but need not be, infinite in either direction. The line has evenly spaced vertices along it.

3.1. **Weight diagrams.** A *weight* λ is a choice of symbols $\times, \circ, \vee, \wedge$ for each vertex, such that $\vee$ only appears finitely many times. Let $\leq$ be the partial (Bruhat) order generated by the condition $\lambda \leq \mu$ whenever μ is gotten from λ by swapping a $\vee$ with a $\wedge$, so that the $\vee$ moves to the right. These need not be adjacent. Write $\sim$ for the equivalence relation generated by $\leq$. A *block* Λ is an equivalence class of weights.

3.2. **Cup and cap diagrams.** A *cup diagram* a is a diagram of non-crossing rays and cups emanating down from the number line, ending on vertices (not necessarily all vertices). If a vertex does not touch a cup or ray, we call it a *free vertex.* A *cap diagram* b is a diagram such that its mirror image across the number line is a cup diagram. For a cup or cap diagram c, we write c^* for its mirror image, which becomes a cap or cup diagram.

Given a cup diagram a and weight λ, we may superimpose them both on the number line, and write it as $a\lambda$. We say $a\lambda$ is an *oriented cup diagram* (OCuD) if:

(a) The free vertices of a are exactly vertices with $\times$ or $\circ$;
(b) A cup has one vertex a $\vee$ and the other a $\wedge$ (so cups are 'oriented');
(c) There cannot be two rays from vertices $\vee\wedge$ in that order.

Given a cap diagram b and weight λ we may superimpose them to form λb. We say λb is an *oriented cap diagram* (OCaD) is $b^*\lambda$ is an OCuD.

Finally, given a cup diagram a, cap diagram b, and weight λ, we may superimpose them to form $a\lambda b$. We say $a\lambda b$ is an oriented circle diagram (OCiD) if $a\lambda$ is an OCuD and λb is an OCaD.

We define the *degree* of an oriented cup, cap, or circle diagram to be the number of clockwise caps and cups in it.

3.3. **Diagrams $\underline{\lambda}$ and $\overline{\lambda}$.** Given a weight diagram λ, there exists unique cup and cap diagrams $\underline{\lambda}$ and $\overline{\lambda}$ such that $\deg(\underline{\lambda}\lambda) = \deg(\lambda\overline{\lambda}) = \deg(\underline{\lambda}\lambda\overline{\lambda}) = 0$. Further we have $\underline{\lambda}^* = \overline{\lambda}$.

The proof of their existence is not so hard, and to construct $\underline{\lambda}$ one draws a cup between adjacent vertices with $\vee$ followed by $\wedge$ until there are no more such pairs left, and then one draws rays emanating from the rest.

Definition 3.1. We define the *defect* of a weight λ to be the number of cirles in $\underline{\lambda}\lambda\overline{\lambda}$.

Lemma 3.2. *If $\lambda \sim \mu$, then $\underline{\lambda} = \underline{\mu}$ if and only if $\lambda = \mu$. Further, every OCuD $a\lambda$ is of the form $\underline{\mu}\lambda$ for some weight μ such that $\lambda \leq \mu$. Therefore, all OCiD's with underlying weight λ are of the form $\underline{\mu}\lambda\overline{\nu}$, with $\mu, \nu \leq \lambda$.*

We will from here on out write $\mu \subseteq \lambda$ if $\underline{\mu}\lambda$ is an OCuD, or $\lambda \supseteq \mu$ if $\lambda\overline{\mu}$ is an OCaD.

3.4. **Khovanov's (generalized) diagram algebra.** Let Λ be a block. We define the Khovanov algebra for the block Λ to be the graded algebra K_Λ with underlying graded basis

$$\{a\lambda b \text{ an OCiD with } \lambda \in \Lambda\} = \{\underline{\mu}\lambda\overline{\nu} : \mu, \lambda, \nu \in \Lambda, \mu \subseteq \lambda \supseteq \nu\}$$

where the degree of $a\lambda b$ is its degree as a OCiD.

In particular, $(K_\Lambda)_0 = \text{span}\{\underline{\lambda}\lambda\overline{\lambda} : \lambda \in \Lambda\}$. We write $e_\lambda = \underline{\lambda}\lambda\overline{\lambda}$.

The multiplication is given by a certain 'surgery' procedure, which was originally defined by Khovanov via a 2-dimensional TQFT, although in a slightly less general context.

In words, the procedure is as follows. First we define $(a\lambda b)(c\mu d) = 0$ if $c^* \neq b$. When $c^* = b$, put the OCiD $a\lambda b$ beneath $c\mu d$ and connect all rays. Then, wherever a cup of $c\mu d$ and cap of $a\lambda d$ are opposite one another, and have no cup or cap between them, we cut each open and glue the ends together. Then the result is a sum of diagrams, where the weights need to be determined according to certain rules. They are as follows. Write 1 for a counter-clockwise circle, x for a clockwise circle, and y for a line. Then the rules are:

$$1 \otimes 1 \mapsto 1, \quad 1 \otimes x \mapsto x, \quad x \otimes x \mapsto 0$$

$$1 \otimes y \mapsto y, \quad x \otimes y \mapsto 0 \quad y \otimes y \mapsto 0$$

$$1 \mapsto x \otimes 1 + 1 \otimes x, \quad x \mapsto x \otimes x, \quad y \mapsto y \otimes x$$

These rules should be understood as, e.g. for $1 \otimes x \mapsto x$, we are saying the if we perform a surgery where we start with a clockwise circle and counterclockwise circle and get a circle, we should orient the new circle clockwise. Or the rule $1 \mapsto x \otimes 1 + 1 \otimes x$ means if after surgery we have turned a counterclockwise circle into two circles, we should take a sum of two diagrams, one where one circle is clockwise and the other counterclockwise, and vice-versa.

Example 3.3. Consider the number line with two vertices and the block Λ with weights $\lambda = \wedge\vee$ and $\mu = \vee\wedge$. The algebra K_Λ is also written as K_1^1. We have

$$K_1^1 = \mathbb{C}\langle e_\lambda, e_\mu \rangle \oplus \mathbb{C}\langle \underline{\lambda}\lambda\overline{\mu}, \underline{\mu}\lambda\overline{\lambda} \rangle \oplus \mathbb{C}\langle \underline{\mu}\lambda\overline{\mu} \rangle$$

where the above summands are the degree 0, 1, and 2 components of K_1^1. Some examples of products in K_1^1:

$$e_\lambda^2 = e_\lambda, \quad e_\mu^2 = e_\mu$$

$$(\underline{\lambda}\lambda\overline{\mu})(\underline{\mu}\lambda\overline{\lambda}) = 0, \quad (\underline{\mu}\lambda\overline{\lambda})(\underline{\lambda}\lambda\overline{\mu}) = \underline{\mu}\lambda\overline{\mu}$$

A full multiplication table for this algebra is given in [BS11a]. Note that the category of finite-dimensional modules over K_1^1 is equivalent to the principal block of category $\mathcal{O}$ for $\mathfrak{sl}_2$.

Example 3.4. More generally, if we take a number line with $n+m$ vertices we define K_m^n to be the algebra K_Λ where Λ is the block with n $\wedge$'s and m $\vee$'s. Then K_m^n is a finite-dimensional Koszul quasi-hereditary algebra (i.e. it is *really* nice) and the category of finite-dimensional modules over it is equivalent to the principal block of parabolic category $\mathcal{O}$ for the pair $(\mathfrak{gl}(m+n), \mathfrak{gl}(m) \oplus \mathfrak{gl}(n))$. This is the main result of [BS11b]. We will most likely not have time to discuss this further.

Example 3.5. If we take a number line which is unbounded in both directions and let Λ be the block with r $\vee$'s and the rest of the vertices $\wedge$'s we get the algebra K_r^∞. This algebra is also isomorphic to the direct limit of the algebras K_r^n where we let n go to ∞.

The main result of [BS09] is that the category of finite-dimensional left K_r^∞ modules is equivalent to the principal block of $Rep(GL(m|n))$, i.e. the block containing the trivial module.

Before discussing further the result in the above example, we state some general results about the Khovanov diagram algebras defined above.

Proposition 3.6. *Let Λ be a block. The following hold:*

(a) *We have*

$$K_\Lambda = \bigoplus_{\lambda,\mu\in\Lambda} e_\lambda K_\lambda e_\mu$$

We describe this property as saying that K_Λ is locally unital.

(b) $e_\lambda e_\mu = \delta_{\lambda\mu} e_\lambda$.

(c) *The map $a\lambda b \mapsto b^*\lambda a^*$ is an algebra anti-automorphism.*

(d) *Given two OCiD's $a\lambda b$, $c\mu d$, we have: $(a\lambda b)(c\mu d) = 0$ unless $b = c^*$; if $b = c^*$ and $a\mu$ is an OCuD, we can write*

$$(a\lambda b)(c\mu d) = s_{a\lambda b}(\mu) a\mu d + (\dagger)$$

and if $a\mu$ is not an OCuD, we have

$$(a\lambda b)(c\mu d) = (\dagger)$$

where in both cases of the above, $(\dagger)$ is a linear combination of elements $a\nu d$, where $\lambda, \mu \leq \nu$. In the first case, $s_{a\lambda b}(\mu) \in \{0, 1\}$, and does not depend on d.

We consider the category $Rep(K_\lambda)$, i.e. finite-dimensional graded left K_λ-modules M such that

$$M = \bigoplus_{\lambda\in\Lambda} e_\lambda M$$

The above results essentially prove that $Rep(K_\Lambda)$ is graded *cellular*, a notion developed originally for algebras in [GL96], which is a bit weaker than being a highest weight category. However, $Rep(K_\lambda)$ is in fact also a highest weight category, with poset Λ with its natural ordering. We describe the irreducible, projective, and standard objects now:

Simples objects: We notice that

$$K_\Lambda/(K_\Lambda)_{>0} = \bigoplus_{\lambda\in\Lambda} \mathbb{C}\langle e_\lambda\rangle$$

it follows that we have one simple object $\mathcal{L}(\lambda)$ spanned by e_λ in the above quotient of pure degree 0. The shifts of the $\mathcal{L}(\lambda)$ give all simple modules in $Rep(K_\Lambda)$.

Projective objects: The projective cover of $\mathcal{L}(\lambda)$ is

$$\mathcal{P}(\lambda) = K_\Lambda e_\lambda$$

This module has natural basis given by

$$\{\underline{\nu}\mu\overline{\lambda} \text{ an OCiD}\}$$

By abstract nonsense, it is a fact that

$$\mathrm{Hom}(\mathcal{P}_\lambda, \mathcal{P}_\mu) = e_\lambda K_\Lambda e_\mu$$

and therefore this Hom space has natural basis

$$\{\underline{\lambda}\nu\overline{\mu} : \lambda \subseteq \nu \supseteq \mu\}$$

Define the q-*Cartan matrix* of Λ to be $C_\Lambda(q) = (c_{\lambda\mu}(q))_{\lambda,\mu\in\Lambda}$, where

$$c_{\lambda\mu}(q) = \sum_{i\geq 0} q^i \dim \mathrm{Hom}(\mathcal{P}(\lambda), \mathcal{P}(\mu))_i$$

This polynomial records the graded multiplicities of $\mathcal{L}(\lambda)$ in $\mathcal{P}(\mu)$. Then we have shown that

$$c_{\lambda\mu}(q) = \sum_{\lambda\subseteq\nu\supseteq\mu} q^{\deg \underline{\lambda}\nu\overline{\mu}}$$

In particular, C_Λ is upper triangular with diagonal entries which are invertible over $\mathbb{Z}[\![q]\!]$.

We also observe that,

$$\mathrm{End}(\mathcal{P}_\lambda) = \mathbb{C}\langle \underline{\lambda}\mu\overline{\lambda} \text{ an OCiD}\rangle$$

One can show that the number of μ such that $\underline{\lambda}\mu\overline{\lambda}$ is an OCiD is $2^{def(\lambda)}$, where $def(\lambda)$ is the number of circles in $\underline{\lambda}\lambda\overline{\lambda}$. In fact in general we have

$$\mathrm{End}(\mathcal{P}(\lambda)) \cong (\mathbb{C}[x]/x^2)^{\otimes def(\lambda)}$$

And this is not so difficult to compute.

Standard modules

For a weight λ, define the standard module $\mathcal{V}(\lambda)$ to be the graded left K_Λ-module with homogeneous basis all OCuD's $\underline{\mu}\lambda$ for $\mu \in \Lambda$, or equivalently all OCuD's $c\lambda$. Then this module is finite-dimensional under our assumptions. We define the action by setting $(a\mu b)(c\lambda) = 0$ if $b \neq c^*$ or $a\lambda$ is not oriented; and if $b = c^*$ and $a\lambda$ is oriented, define

$$(a\mu b)(c\lambda) = s_{a\mu b}(\lambda) a\lambda$$

where the definition of $s_{a\mu b}(\lambda)$ was given in proposition 3.6. This gives $\mathcal{V}(\lambda)$ a well-defined structure of a K_Λ-module. The module $\mathcal{V}(\lambda)$ has the nice property of being *rigid*- its socle and radical filtrations agree. Further, the ith layer of the radical filtration has natural basis all OCuD's $a\lambda$ of degree i.

3.5. **Quasi-hereditariness.** Using proposition 3.6 one can show that:

Proposition 3.7. *The module $\mathcal{P}(\lambda)$ admits a filtration*

$$\{0\} = M(0) \subseteq M(1) \subseteq \cdots M(2^{def(\lambda)}) = \mathcal{P}(\lambda)$$

such that

$$M(i)/M(i-1) = \mathcal{V}(\mu_i)\langle \deg(\mu_i\overline{\lambda})\rangle, \text{ where } \lambda \subseteq \mu_i$$

Further, the μ_i may be chosen so that if $\mu_i > \mu_j$ then $i < j$. The notation $N\langle - \rangle$ denotes a grading shift of the module N.

The above proposition is essential in proving that $Rep(K_\Lambda)$ is a (positively graded) highest weight category (with duality).

Now we may define another matrix D_Λ with entries $(d_{\lambda\mu}(q))_{\lambda,\mu}$, where

$$d_{\lambda\mu}(q) = q^{\deg \lambda\mu} \text{ if } \lambda \subseteq \mu, \text{ otherwise } = 0$$

Then in fact $d_{\lambda\mu}$ records the graded multiplicity of $L(\lambda)$ in $V(\mu)$ as well as the graded multiplicity of $V(\mu)$ in $P(\lambda)$ (with respect to the filtration by standard modules in the above), by BGG reciprocity. We also have $C_\Lambda(q) = D_\Lambda(q)D_\Lambda(q)^T$ by BGG reciprocity.

3.6. **Koszulity.** Another remarkable property of the algebra K_Λ is that it is a *locally unital Koszul algebra*; by this we mean it is positively graded, $K/K_{>0}$ is a direct sum of simple algebras, and $K/K_{>0}$ has a graded projective resolution of form

$$\cdots \to P_2 \to P_1 \to P_0 \to K/K_{>0} \to 0$$

where each P_i is projective and generated in degree i. If $|\Lambda| < \infty$ this in particular implies that K_Λ is a Koszul algebra. In general, it also implies that K_λ is a *quadratic* algebra, i.e. it is generated in degree one and its relations are generated by homogeneous elements of degree 2.

3.7. **Relation to $Rep(GL(m|n))$.** For each dominant integral weight $\lambda \in P^+$, associate the sequence $(c_1, \ldots, c_m | d_1, \ldots, d_n)$, where $c_i = (\lambda + \rho, \epsilon_i)$, $d_i = (\lambda + \rho, \delta_i)$. Given such a sequence, we construct its weight diagram, as follows: we use the number line which is unbounded in both directions, and label the vertices with integers.

Write $I = \{c_1, \ldots, c_m, d_1, \ldots, d_n\}$. Then if $n \notin I$, we put a $\wedge$ at n. If $n \in I$, and $n = c_i = d_j$ for some i and j, we put a $\vee$ at n. If $n = c_i$ but $n \neq d_j$ for any j, we put a $\times$, and if $n = d_i$ but $n \neq c_j$ for any j, we put a $\circ$.

We will denote the weight diagram of λ by λ once again. Notice that all but finitely many vertices of λ are $\wedge$'s. We define the *degree of atypicality* of λ to be the number of $\vee$'s appearing in its weight diagram. This is also the defect of the block containing λ. We say λ is typical if its degree of atypicality is zero.

Lemma 3.8. *If $\lambda \in P^+$, then the weights in the block containing $L(\lambda)$ are exactly the weights in the block Λ consisting of weight diagrams equivalent to λ.*

Write $K(m|n) = \bigoplus_{\Lambda} K_\Lambda$, where the sum runs over all blocks Λ with weights coming from $Rep(GL(m|n))$ via the above procedure. Write $Mof(K(m|n))$ for the category of finite-dimensional left $K(m|n)$-modules.

The main theorem is:

Theorem 3.9. *There is an equivalence of categories $\mathbb{E} : Mof(K(m|n)) \to Rep(GL(m|n))$ such that $\mathbb{E}(\mathcal{L}(\lambda)) = L(\lambda)$, $\mathbb{E}(\mathcal{V}(\lambda)) = K(\lambda)$, and $\mathbb{E}(\mathcal{P}(\lambda)) = P(\lambda)$. In fact, we have*

$$K(m|n) \cong \mathrm{End}^{fin}_{\mathfrak{g}}(\bigoplus_{\lambda \in P^+} P(\lambda)) := \bigoplus_{\lambda,\mu \in P^+} \mathrm{Hom}_g(P(\lambda), P(\mu))$$

Note- the above equvialence does *not* preserve the tensor product structure- in particular we see that the tensor product of simple $K(m|n)$-modules is simple, while the same is not true for simple modules in $Rep(GL(m|n))$.

Corollary 3.10. (a) *The blocks of $Rep(GL(m|n))$ are naturally Koszul.*

(b) *Kac modules in $Rep(GL(m|n))$ are rigid- that is, their socle and radical filtrations agree.*

(c) *If $P(\lambda)$ is the projective cover of $L(\lambda)$ in $Rep(GL(m|n)))$, we have*

$$\mathrm{End}_{\mathfrak{g}}(P(\lambda)) \cong (\mathbb{C}[x]/x^2)^{\otimes r}$$

where r is the degree of atypicality of λ.

(d) *A block of atypicality degree r in $Rep(GL(m|n))$ is equivalent to the principal block of $Rep(GL(r|r))$.*

These facts follow immediately from the analogous statements for $Mof(K(m|n))$.

Example 3.11. Let us return to the case of *Rep*$(GL(1|1))$, and understand the principal block using our connection with Khovanov algebras. By the theorem, it is equivalent to left K_1^∞-modules. The weight $\lambda_n = n(\epsilon_1 - \delta_1)$ will have a $\vee$ at the nth vertex, and a wedge everywhere else.

We see that module $\mathcal{V}(\lambda_n)$ has basis

$$\{\underline{\lambda_n}\lambda_n, \underline{\lambda_{n-1}}\lambda_n\}$$

with layers:

$\underline{\lambda_n}\lambda_n$
$\underline{\lambda_{n-1}}\lambda_n$

The projective cover $\mathcal{P}(\lambda_n)$ has basis

$$\{\underline{\lambda_n}\lambda_n\overline{\lambda_n}, \underline{\lambda_{n-1}}\lambda_n\overline{\lambda_n}, \underline{\lambda_n}\lambda_{n+1}\overline{\lambda_n}, \underline{\lambda_{n+1}}\lambda_{n+1}\overline{\lambda_n}\}$$

with layers:

$\underline{\lambda_n}\lambda_n\overline{\lambda_n}$
$\underline{\lambda_{n-1}}\lambda_n\overline{\lambda_n} \oplus \underline{\lambda_{n+1}}\lambda_{n+1}\overline{\lambda_n}$
$\underline{\lambda_n}\lambda_{n+1}\overline{\lambda_n}$

Hence we see that $\mathcal{P}(\lambda_n)$ has head and tail isomorphic to $\mathcal{L}(\lambda_n)$, and

$$\operatorname{rad}\mathcal{P}(\lambda_n)/\operatorname{soc}\mathcal{P}(\lambda_n) \cong \mathcal{L}(\lambda_{n-1}) \oplus \mathcal{L}(\lambda_{n+1}).$$

Hence $\operatorname{End}_{\mathfrak{g}}^{fin}(\bigoplus_{n\in\mathbb{Z}} P(\lambda_n))$ has a natural basis given by $\{e_n, a_n, b_n, c_n : n \in \mathbb{Z}\}$, where e_n is the projection onto $P(\lambda_n)$, and the a_n, b_n, and c_n may be chosen so that:

$$a_n : P(\lambda_n) \to P(\lambda_{n+1}), \quad b_n : P(\lambda_{n+1}) \to P(\lambda_n), \quad c_n := b_n \circ a_n = a_{n-1}b_{n-1}$$

Therefore this is again equivalent to the infinite quiver described before. Further, we notice that the quiver has all relations homogeneous of degree 2- this is a consequence of Koszulity. Indeed, the natural grading by path length on the path algebra will agree with the Koszul grading on K_Λ.

Example 3.12. If we consider *Rep*$(GL(1|n))$ for any n (or *Rep*$(GL(n|1))$), all blocks of $K(1|n)$ will be equivalent to either K_1^∞ or $K_0^\infty \cong \mathbb{C}$. Therefore our analysis in the $GL(1|1)$ case tells us the structure of all blocks for $GL(1|n)$.

Rough idea of proof of main theorem: Brundan and Stroppel consider, for positive integers p, q, the weight

$$\lambda_{p,q} = \sum_{i=1}^{m} p\epsilon_i - \sum_{j=1}^{n}(j+m)\delta_j$$

By construction, $\lambda_{p,q}$ is a typical weight, and so $L(\lambda_{p,q}) = K(\lambda_{p,q}) = P(\lambda_{p,q})$. It follows that for any $d \geq 0$, $L(\lambda_{p,q}) \otimes V^{\otimes d}$ is projective, where $V = \mathbb{C}^{m|n}$ is the standard module for $\mathfrak{gl}(m|n)$. Then every projective indecomposable $P(\lambda)$ in *Rep*$(GL(m|n))$ shows up in such a tensor product for some $p, q, d \geq 0$. The goal is to understand the endomorphism algebra of this object.

An action is constructed of the degenerate affine Hecke algebra H_d for S_d, i.e. the algebra $\mathbb{C}[x_1, \dots, x_n] \otimes \mathbb{C}[S_n]$, where $\mathbb{C}[x_1, \dots, x_d]$ and $\mathbb{C}[S_n]$ are subalgebras and we have relations $s_r x_s = x_s s_r$ if $s \neq r, r+1$, $s_r x_{r+1} = x_r s_r + 1$. The actions is given by letting S_d act by permutations on $V^{\otimes d}$, and the polynomial generators x_s act by certain comultiples and tensors of the casimir on the $V^{\otimes d}$ factors.

The remarkable fact is that this induces a surjection of H_d onto $\mathrm{End}_{\mathfrak{g}}(L(\lambda_{p,q} \otimes V^{\otimes d})$, giving a Schur-Weyl duality-like correspondence between $Rep(GL(m|n))$ and the image of H_d. Then after a lot of work they are able to construct a Morita equvialence between the image of H_d and a certain Khovanov algebra. Taking direct limits gives the result.

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