REPRESENTATIONS OF THE GENERAL LINEAR SUPERGROUP

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The subject of this talk is to relate certain *generalized Khovanov diagram algebras* to the representation theory of the general linear supergroup $GL(m|n)$. The talk is based on a series of four papers by Brundan and Stroppel where they do much more than this: [BS11a], [BS09],[BS11b], and [BS10]. In order to motivate this connection, we discuss some basic aspects of representation theory in the 'super' world. However, given the time constraints, we will skip over many topics and points, and few proofs can be discussed. The interested listener can read more about representations of lie superalgebras from Serganova's survey in [CCC+17] (pp. 125-177), or the books [CW12], [Mus12].

All vector spaces and algebras in this talk are over $\mathbb{C}$, the complex numbers (for simplicity). As always, please be careful, with what you read, as there are possibly errors in what is written. See references for further details.

1. The Category $SVect$; Lie Superalgebras

We start by defining the symmetric monoidal category $SVect$. The objects, which we call *super vector spaces*, are $\mathbb{Z}_2$-graded vector spaces $V = V_{\overline{0}} \oplus V_{\overline{1}}$. For a homogeneous element $x \in V$, we write $\overline{x} \in \mathbb{Z}_2$ for its degree. The morphisms are degree-preserving maps of vector spaces. The monoidal structure is given by $\mathbb{Z}_2$-graded tensor product: for V, W super vector spaces, we define the super vector space $V \otimes W$ with grading:

$$(V \otimes W)_{\overline{0}} = V_{\overline{0}} \otimes W_{\overline{0}} \oplus V_{\overline{1}} \otimes W_{\overline{1}}, \quad (V \otimes W)_{\overline{1}} = V_{\overline{1}} \otimes W_{\overline{0}} \oplus V_{\overline{0}} \otimes W_{\overline{1}}$$

Then the object $\mathbb{C} := \mathbb{C}^{1|0} = \mathbb{C} \oplus 0$ is the unit object. So far nothing of interest has happened. To enter the super world, we take the following braiding isomorphism:

$$\sigma_{VW} : V \otimes W \xrightarrow{\sim} W \otimes V, \qquad v \otimes w \mapsto (-1)^{\overline{v}\overline{w}} w \otimes v$$

Remark 1.1.
There is a parity shift functor $\Pi : SVect \to SVect$ defined by $(\Pi V)_{\overline{i}} = V_{\overline{i+1}}$, and given $f : V \to W$, $\Pi(f)$ is f once again but on the parity shifted spaces.

Since $\sigma_{WV} \circ \sigma_{VW} = \mathrm{id}_{V \otimes W}$, we get an induced action of S_n on $V^{\otimes n}$. Hence any construction we can do classically that can be phrased in terms of representations of S_n, we can do here. In particular, all Schur functors $S_\lambda(V)$ (e.g. symmetric or exterior powers) are defined for a super vector space V (althought they are *not* the same as the classical Schur functors in the category of vector spaces with the usual braiding.)

In particular, we may define a lie algebra in $SVect$ to be an object $\mathfrak{g}$ with a map $[-,-] : \Lambda^2\mathfrak{g} \to \mathfrak{g}$ such that $[-,[-,-]] : \mathfrak{g}^{\otimes 3} \to \mathfrak{g}$ is killed by $(1 + \sigma + \sigma^2) \in \mathbb{C}[S_3]$, where $\sigma \in S_3$ is a three-cycle (this is the *super* Jacobi identity). We call a lie algebra in the category of $SVect$ a *Lie superalgebra*.

Some observations: if $\mathfrak{g}$ is a Lie superalgebra, then $\mathfrak{g}_{\bar{0}}$ is a Lie algebra, $\mathfrak{g}_{\bar{1}}$ is a $\mathfrak{g}_{\bar{0}}$-module, and the commutator defines a map $[-,-] : S^2\mathfrak{g}_{\bar{1}} \to \mathfrak{g}_{\bar{0}}$ which is $\mathfrak{g}_{\bar{0}}$-equivariant (here $S^2\mathfrak{g}_{\bar{1}}$ is the usual second symmetric power).

An example: given a superalgebra $A = A_{\bar{0}} \oplus A_{\bar{1}}$, we can construct a Lie superalgebra from it via its commutator in $SVect$ (otherwise know as the *supercommutator*), which can be computed to be:

$$[a, b] = ab - (-1)^{\bar{a}\bar{b}}ba$$

Given a super vector space V, we may define $\underline{\mathrm{End}}(V)$ to be all vector space endomorphisms of V (not necessarily of degree 0). Then $\underline{End}(V)$ is a superalgebra, where the degree $\bar{i}$ part consists of maps $V \to V$ of degree $\bar{i}$. The lie superalgebra $\mathfrak{gl}(V)$ is one given by taking $\underline{End}(V)$ with supercommutator. We call $\mathfrak{gl}(V)$ the general linear superalgebra from V. In particular, we define $\mathfrak{gl}(m|n) := \mathfrak{gl}(\mathbb{C}^{m|n})$, where $\mathbb{C}^{m|n} = \mathbb{C}^m \oplus \mathbb{C}^n$.

A representation of a Lie superalgebra $\mathfrak{g}$ is a super vector space V with a Lie superalgebra homomorphism $\mathfrak{g} \to \mathfrak{gl}(V)$. We also may define the enveloping algebra $\mathcal{U}\mathfrak{g}$ as the superalgebra $T^\bullet\mathfrak{g}/\langle [x, y] - (xy - (-1)^{\bar{x}\bar{y}}yx)\rangle$. As usual, representations of $\mathfrak{g}$ are equivalent to modules over its enveloping algebra. Further, there is a PBW theorem for superalgebras, and an important consequence of it is that $\mathcal{U}\mathfrak{g}$ is a free $\mathcal{U}\mathfrak{g}_0$-module of finite rank $2^{\dim \mathfrak{g}_{\bar{1}}}$.

2. Representations of The General Linear Supergroup

We now discuss $\mathfrak{g} = \mathfrak{gl}(m|n)$ further. It is presented by matrices of the form

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$$

where $\mathfrak{g}_{\bar{0}}$ consists of matrices of the form:

$$\left[\begin{array}{c|c} A & 0 \\ \hline 0 & D \end{array}\right]$$

and we observe $\mathfrak{g}_{\bar{0}} \cong \mathfrak{gl}(m) \times \mathfrak{gl}(n)$ as a Lie algebra. Also, $\mathfrak{g}_{\bar{1}}$ consists of matrices of the form:

$$\left[\begin{array}{c|c} 0 & B \\ \hline C & 0 \end{array}\right]$$

which is isomorphic to the $\mathfrak{g}_{\bar{0}}$-module $\mathbb{C}^n \otimes (\mathbb{C}^m)^* \oplus (\mathbb{C}^n)^* \otimes \mathbb{C}^m$.

In fact, $\mathfrak{g}$ has a nice $\mathbb{Z}$-grading: $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$, where $\mathfrak{g}_0 = \mathfrak{g}_{\bar{0}}$, and

$$\mathfrak{g}_1 = \left[\begin{array}{c|c} 0 & B \\ \hline 0 & 0 \end{array}\right], \quad \mathfrak{g}_{-1} = \left[\begin{array}{c|c} 0 & 0 \\ \hline C & 0 \end{array}\right]$$

Goal: We are interested in understanding the category of finite-dimensional $\mathfrak{g}$-modules which are integrable over $\mathfrak{g}_0$. We call this category *Rep*$(GL(m|n))$. We do not discuss $GL(m|n)$ explicitly, but its finite-dimensional representation theory is equivalent *Rep*$(GL(m|n))$ as we have defined it (via the theory of super Harish-Chandra pairs).

Some remarks:

(1) *Rep*$(GL(m|n))$ is *not* a semisimple category. However all objects have finite-length.
(2) *Rep*$(GL(m|n))$ has a highest weight theory similar to that of semisimple Lie algebras.

(3) In fact, *Rep*$(GL(m|n))$ is a so-called *highest weight category* in the sense of Cline, Parshall, and Scott (see [CPS88]). This underlies many of the ideas of this talk.
(4) The simple objects of *Rep*$(GL(m|n))$ (up to parity) are in (reasonably natural) bijection with the simple objects of *Rep*$(GL(m) \times GL(n))$.
(5) *Rep*$(GL(m|n))$ has a enough projectives- the functor $\mathrm{Ind}_{\mathfrak{g}_{\bar{0}}}^{g}(-) = \mathcal{U}\mathfrak{g} \otimes_{\mathcal{U}\mathfrak{g}_{\bar{0}}} (-)$ is exact and left adjoint to restriction, and therefore takes $\mathfrak{g}_{\bar{0}}$-modules to projectives.
(6) An object in *Rep*$(GL(m|n))$ is projective if and only if it is injective. Therefore projectives admit no non-trivial extensions with any object.

Now we setup some notation to get specific. Choose $\mathfrak{h}$ to be the subalgebra of all diagonal matrices, and write $\epsilon_1, \dots, \epsilon_m, \delta_1, \dots, \delta_n$ for the usual basis of $\mathfrak{h}^*$ (here ϵ_i is dual to the ith diagonal entry and δ_i is dual to the $(m+i)$th diagonal entry). Define

$$\rho = -\epsilon_2 - 2\epsilon_3 - \cdots - (m-1)\epsilon_m + (m-1)\delta_1 + \cdots + (m-n)\delta_n$$

(here ρ is (roughly) the half sum of all even roots minus the half sum of all odd roots). We have inner product on $\mathfrak{h}^*$ given by

$$(\epsilon_i, \epsilon_j) = \delta_{ij}, \quad (\epsilon_i, \delta_j) = 0, \quad (\delta_i, \delta_j) = -\delta_{ij}$$

Write $P = \mathbb{Z}\langle \epsilon_1, \dots, \epsilon_m, \delta_1, \dots, \delta_n \rangle$ for the collection of integral weights, P^+ for the integral weights which are dominant over $\mathfrak{g}_0$, which we recall consists of all weights of the form:

$$\lambda = a_1\epsilon_1 + \cdots + a_m\epsilon_m + b_1\delta_1 + \cdots + b_n\delta_n, \quad \text{where } a_1 \geq a_2 \geq \cdots \geq a_m, \;\; b_1 \geq b_2 \geq \cdots \geq b_n$$

equivalently

$$(\lambda+\rho, \epsilon_1) > (\lambda+\rho, \epsilon_2) > \cdots > (\lambda+\rho, \epsilon_m), \quad (\lambda+\rho, \delta_1) < (\lambda+\rho, \delta_2) < \cdots < (\lambda+\rho, \delta_n)$$

Now, let $\lambda \in P^+$ be a dominant integral weight. Then we may construct the finite-dimensional $\mathfrak{g}_{\bar{0}}$-module $L_0(\lambda)$, irreducible of highest weight λ. Extend $L_0(\lambda)$ to a $\mathfrak{g}_0 \oplus \mathfrak{g}_1$-module by letting $\mathfrak{g}_1$ act by 0. Then we define the *Kac*-module as

$$K(\lambda) := \mathrm{Ind}_{\mathfrak{g}_0 \oplus \mathfrak{g}_1}^{\mathfrak{g}} L_0(\lambda) = \mathcal{U}\mathfrak{g} \otimes_{\mathcal{U}(\mathfrak{g}_0 \oplus \mathfrak{g}_1)} L_0(\lambda).$$

By the PBW theorem mentioned above, $K(\lambda)$ is finite-dimensional. It turns out that $K(\lambda)$ is indecomposable and has a unique irreducible finite-dimensional quotient, which we call $L(\lambda)$. Further, every finite-dimensional quotient is isomorphic to $L(\lambda)$ for some $\lambda \in P^+$ (up to parity shift).

Write $P(\lambda)$ for projective cover of $L(\lambda)$- that is, an indecomposable projective such that it's largest semisimple quotient is $L(\lambda)$. We have a sequence of surjections:

$$P(\lambda) \twoheadrightarrow K(\lambda) \twoheadrightarrow L(\lambda)$$

When we have $P(\lambda) = K(\lambda) = L(\lambda)$ we say that λ is a *typical* weight. There is a numerical condition that is equivalent to typicality, to be discussed below.

Our goal is to understand the category *Rep*$(GL(m|n))$. For this, we look at the algebra

$$K = \mathrm{End}_{\mathfrak{g}}\Big(\bigoplus_{\lambda \in P^+} P(\lambda)\Big) \cong \bigoplus_{\lambda, \mu \in P^+} \mathrm{Hom}_{\mathfrak{g}}(P(\lambda), P(\mu))$$

(where the isomorphism comes from $\mathrm{Hom}_{\mathfrak{g}}(P(\lambda), P(\mu)) = 0$ for all but finitely many μ if we fix λ). Write $Mof(K)$ for the category of finite-dimensional K-modules. Then by some abstract nonsense, we have an equivalence of categories

$$Rep(GL(m|n)) \xrightarrow{\sim} Mof(K)$$

a representation V goes to $\mathrm{Hom}_{\mathfrak{g}}(\bigoplus_{\lambda \in P^+} P(\lambda), V)$.

Therefore, we would like to understand $\mathrm{Hom}_{\mathfrak{g}}(P(\lambda), P(\mu))$ for arbitrary dominant weights λ, μ.

Example 2.1. Here we discuss the case $Rep(GL(1|1))$. In $\mathfrak{gl}(1|1)$ we have the element $h = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ which is central. Hence its action separates representations. One can show that h acts by a non-zero scalar on $L(\lambda)$, then λ is typical, i.e. $L(\lambda)$ is projective, and hence has no non-trivial extensions with other objects.

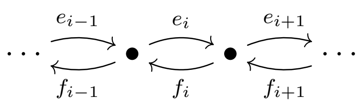
Write $\mathcal{O}_0$ for the subcategory of $Rep(GL(1|1))$ where h acts by 0. This is referred to as the 'principal block' of $Rep(GL(1|1))$. The irreducibles are $L(n) := L(n(\epsilon_1 - \delta_1))$, where $n \in \mathbb{Z}$, along with their parity shifts. These are one-dimensional modules and we concentrate them in degree $n(\text{mod } 2)$. The Kac module $K(n)$ and projective $P(n)$ look like:

$$K(n) = \begin{array}{|c|}\hline L(n) \\ \hline L(n-1) \\ \hline \end{array} \qquad P(n) = \begin{array}{|c|c|}\hline \multicolumn{2}{|c|}{L(n)} \\ \hline L(n-1) & L(n+1) \\ \hline \multicolumn{2}{|c|}{L(n)} \\ \hline \end{array},$$

It follows that

$$\mathrm{End}_{\mathfrak{g}}(P(n)) \cong \mathbb{C}[x]/(x^2) \quad \dim \mathrm{Hom}_{\mathfrak{g}}(P(n), P(n-1)) = \dim \mathrm{Hom}_{\mathfrak{g}}(P(n), P(n+1)) = 1,$$

and $\mathrm{Hom}_{\mathfrak{g}}(P(n), P(m)) = 0$ if $m \neq n-1, n, n+1$. Computing the Ext-quiver, one can show that $Rep(GL(1|1))$ is equivalent to finite-dimensional representations of the infinite quiver



such that $f_i e_i + e_{i-1} f_{i-1} = 0$.

3. Khovanov's Diagram Algebra

In come the diagram algebras. Since I'm not that good at tex-ing, I won't be drawing any pictures here. Watch the talk for some diagrams! Or look at Brundan-Stroppel.

3.1. **Weight Diagrams.** For each dominant integral weight $\lambda \in P^+$, associate the sequence $(c_1, \dots, c_m | d_1, \dots, d_n)$, where $c_i = (\lambda + \rho, \epsilon_i)$, $d_i = (\lambda + \rho, \delta_i)$. Given such a sequence, we construct its *weight diagram*, as follows: it is a number line with a vertex at each integer, where the vertex either looks like:

$$\times, \circ, \vee, \wedge$$

Write $I = \{c_1, \dots, c_m, d_1, \dots, d_n\}$. Then if $n \notin I$, we put a $\wedge$ at n. If $n \in I$, and $n = c_i = d_j$ for some i and j, we put a $\vee$ at n. If $n = c_i$ but $n \neq d_j$ for any j, we put a $\times$, and if $n = d_i$ but $n \neq c_j$ for any j, we put a $\circ$.

We will denote the weight diagram of λ by λ once again. Notice that all but finitely many vertices of λ are $\wedge$'s. We define the *degree of atypicality* of λ to be the number of $\vee$'s appearing in its weight diagram. We say λ is typical if its degree of atypicality is zero.

3.2. **Cup, Cap, and Circle Diagrams.** Define a cup diagram a to be a diagram of non-crossing rays and cups emanating down from the number line, ending on vertices (not necessarily all vertices). A cap diagram b is a diagram such that its mirror image across the number line is a cap diagram. Given a cup diagram a and weight λ, we may superimpose them both on the number line, and write it as $a\lambda$.

We say that $a\lambda$ is an oriented cup diagram (OCuD) if: all rays have $\wedge$'s as their source, all cups have one vertex on a $\wedge$ and one on a $\vee$, all vertices with $\wedge$ or $\vee$ are connected to a ray or cup, and no ray or cup is connected to a $\circ$ or $\times$. Similarly given a cap diagram b and a weight λ, we may superimpose them to form λ. We say λb is an oriented cap diagram (OCaD) if, writing b^* for the mirror image of b, $b^*\lambda$ is an OCuD.

Given a cup diagram a, cap diagram b, and weight λ, we may superimpose them all on a number line, and form $a\lambda b$. We say $a\lambda b$ is an oriented circle diagram (OCiD) if both $a\lambda$ is an OCuD and λb is an OCaD.

Define the degree of an OCuD (resp. OCaD) $a\lambda$ (resp. λb) to be the number of clockwise cups (resp. caps) appearing. Then define $\deg(a\lambda b) = \deg(a\lambda) + \deg(\lambda b)$.

3.3. **The Khovanov Algebra.** We may now define the generalized Khovanov algebra $K(m|n)$ (note: this is actually a sum of generalized Khovanov algebras), which is a positively graded algebra, as follows: a homogeneous basis is given by all OCiDs $a\lambda b$ where $\lambda \in P^+$, and the degree of $a\lambda b$ is the degree as defined above. The multiplication is given by a certain 'surgery' procedure, which was originally defined by Khovanov via a 2-dimensional TQFT, although in a slightly less general context.

In words, the procedure is as follows. First we define $(a\lambda b)(c\mu d) = 0$ if $c^* \neq b$. When $c^* = b$, put the OCiD $a\lambda b$ beneath $c\mu d$ and connect all rays. Then, wherever a cup of $c\mu d$ and cap of $a\lambda d$ are opposite one another, and have no cup or cap between them, we cut each open and glue the ends together. Then the result is a sum of diagrams, where the weights need to be determined according to certain rules. They are as follows. Write 1 for a counter-clockwise circle, x for a clockwise circle, and y for a line. Then the rules are:

$$1 \otimes 1 \mapsto 1, \quad 1 \otimes x \mapsto x, \quad x \otimes x \mapsto 0$$

$$1 \otimes y \mapsto y, \quad x \otimes y \mapsto 0 \quad y \otimes y \mapsto 0$$

$$1 \mapsto x \otimes 1 + 1 \otimes x, \quad x \mapsto x \otimes x, \quad y \mapsto y \otimes x$$

These rules should be understood as, e.g. for $1 \otimes x \mapsto x$, we are saying the if we perform a surgery where we start with a clockwise circle and counterclockwise circle and get a circle, we should orient the new circle clockwise. Or the rule $1 \mapsto x \otimes 1 + 1 \otimes x$ means if after surgery we have turned a counterclockwise circle into two circles, we should take a sum of two diagrams, one where one circle is clockwise and the other counterclockwise, and vice-versa.

In this way, the $K(m|n)$ becomes a positively graded algebra. Let us study it.

3.4. **Representation Theory of $K(m|n)$.** First an important fact: given any weight λ, there exists a unique cup diagram $\underline{\lambda}$ and cap diagram $\overline{\lambda}$ such that $\deg(\underline{\lambda}\lambda) = \deg(\lambda\overline{\lambda}) = 0$. The way one constructs $\underline{\lambda}$ for example is to draw cups so that each has counterclockwise orientation with respect to λ. Define $e_\lambda := \underline{\lambda}\lambda\overline{\lambda}$. Then we have

$$K(m|n)_0 = \mathbb{C}\langle e_\lambda \rangle_{\lambda \in P^+}.$$

Further, the e_λ form a system of orthogonal idempotents, and we have

$$K(m|n) = \bigoplus_{\lambda,\mu} e_\lambda K(m|n) e_\mu$$

so $K(m|n)$ is *locally unital*, although it is not unital in general. Further, we have

$$K(m|n)/K_{>0} = \bigoplus_{\lambda \in P^+} \mathbb{C}\langle e_\lambda \rangle$$

where e_λ spans a one-dimensional irreducible submodule of the above quotient. We write this summand as $\mathcal{L}(\lambda)$, and hence the simple left $K(m|n)$-modules are in bijection with P^+.

Now we also have the decomposition, as left $K(m|n)$-modules,

$$K(m|n) = \bigoplus_{\lambda \in P^+} K(m|n) e_\lambda$$

if we write $\mathcal{P}(\lambda) := K(m|n)e_\lambda$, then this is the indecomposable projective cover of $\mathcal{L}(\lambda)$. It has a natural basis

$$\{a\mu\overline{\lambda} \text{ an OCiD}\}$$

Further, we have

$$\mathrm{Hom}_{K(m|n)}(\mathcal{P}(\lambda), \mathcal{P}(\mu)) = \mathrm{Hom}_{K(m|n)}(K(m|n)e_\lambda, K(m|n)e_\mu) \cong e_\lambda K(m|n) e_\mu$$

which has natural basis

$$\{\underline{\lambda}\gamma\overline{\mu} \text{ an OCiD}\}$$

so these Hom spaces are actually too bad to understand. Finally, for $\lambda \in P^+$, we define so-called *cell-modules*, $\mathcal{V}(\lambda)$, which are modules with basis

$$\{(a\lambda| \text{ an OCuD}\}$$

and the action is given by: $(b\mu d)(a\lambda| = 0$ if $d \neq a^*$ or $b\lambda$ is not oriented, and otherwise $(b\mu d)(a\lambda| = s_{b\mu d}(\lambda) b\lambda$, where $a_{b\mu d}(\lambda) \in \{0, 1\}$, and this scalar is 1 iff the product $(b\mu d)(a\lambda\overline{\lambda})$ has $b\lambda\overline{\lambda}$ as a term.

Okay, we successfully bashed through all the definitions we need! We now make some nice further observations:

(1) The module $\mathcal{V}(\lambda)$ is indecomposable and multiplicity-free, and we have surjections $\mathcal{P}(\lambda) \twoheadrightarrow \mathcal{V}(\lambda) \twoheadrightarrow \mathcal{L}(\lambda)$. Further, we can write down an explicit socle filtration of this module and show that it is equal to its radical filtration (i.e. it's *rigid*).
(2) All of $\mathcal{P}(\lambda), \mathcal{V}(\lambda)$, and $\mathcal{L}(\lambda)$ have a natural grading by the degree of the diagram, and in this way they become graded $K(m|n)$-modules, and the surjections in the previous item all respect this grading.

(3) $Mof(K(m|n))$ is a so-called positively graded highest weight category, again in the sense of [CPS88].
(4) The algebra $K(m|n)$ is Koszul, a very nice property of algebras.
(5) One can explicitly compute: $\text{End}_{K(m|n)}(\mathcal{P}(\lambda)) \cong \mathbb{C}[x_1, \ldots, x_r]/(x_1^2, \ldots, x_r^2)$, where r is the number of degree of atypicality of λ.

The main theorem is:

Theorem 3.1. *There is an equivalence of categories* $\mathbb{E} : Mof(K(m|n)) \to Rep(GL(m|n))$ *such that* $\mathbb{E}(\mathcal{L}(\lambda)) = L(\lambda)$, $\mathbb{E}(\mathcal{V}(\lambda)) = K(\lambda)$, *and* $\mathbb{E}(\mathcal{P}(\lambda)) = P(\lambda)$. *In particular,*

$$K(m|n) \cong \text{End}_{\mathfrak{g}}(\bigoplus_{\lambda \in P^+} P(\lambda))$$

References

[BS09] Jonathan Brundan and Catharina Stroppel, *Highest weight categories arising from khovanov's diagram algebra iv: the general linear supergroup*, arXiv preprint arXiv:0907.2543 (2009).

[BS10] ______, *Highest weight categories arising from khovanov's diagram algebra ii: Koszulity*, Transformation groups **15** (2010), no. 1, 1–45.

[BS11a] ______, *Highest weight categories arising from khovanov's diagram algebra i: cellularity*, Moscow Mathematical Journal **11** (2011), no. 4, 685–722.

[BS11b] ______,*Highest weight categories arising from khovanov's diagram algebra iii: Category $\mathcal{O}$* , Representation Theory of the American Mathematical Society **15** (2011), no. 5, 170–243.

[CCC+17] Filippo Callegaro, Giovanna Carnovale, Fabrizio Caselli, Corrado De Concini, and Alberto De Sole, *Perspectives in lie theory*, vol. 19, Springer, 2017.

[CPS88] Edward Cline, Brian Parshall, and Leonard Scott, *Finite dimensional algebras and highest weight categories*, J. reine angew. Math **391** (1988), no. 85-99, 3.

[CW12] Shun-Jen Cheng and Weiqiang Wang, *Dualities and representations of lie superalgebras*, American Mathematical Soc., 2012.

[Mus12] Ian Malcolm Musson, *Lie superalgebras and enveloping algebras*, vol. 131, American Mathematical Soc., 2012.