Making the Jones polynomial more geometric

Calvin McPhail-Snyder

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Acknowledgements

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- Various parts of this research program are due to:
  - C Blanchet
  - KC Chen
  - R Kashaev
  - N Geer
  - S Morrison
  - B Patureau-Mirand
  - N Reshetikhin
  - N Snyder
- In particular, I reference a paper by myself, Chen, Morrison, and Snyder.
- Turaev has been working on some similar ideas, but in a different, more algebraic direction.

Overview

- Quantum invariants like the Jones polynomial are defined in an algebraic way.
- However, there is now a lot of interest in what they say about the geometry of knots and manifolds.
- I want to talk about a research program to address these questions and discuss some examples of these more geometric quantum invariants
- First: a reminder about what I mean by “algebraic”.

Quantum invariants

What is a quantum invariant?

- A knot invariant is a function

$$\{\text{knots}\} \to \text{ numbers, polynomials, etc.}$$

- For our purposes, a quantum invariant is a topological invariant constructed using the representation theory of quantum groups.
- Generally quantum invariants appear as part of topological quantum field theories (TQFTs).

Example: the Jones polynomial

Quantum $\mathfrak{sl}_2$

$\mathcal{U}_q = \mathcal{U}_q(\mathfrak{sl}_2)$ is an algebra over $\mathbb{C}[q, q^{-1}]$ that we can think of as a q-analogue of the universal enveloping algebra of $\mathfrak{sl}_2$.
For q not a root of unity, it acts a lot like $\mathfrak{sl}_2$.
In particular, there is one[1] representation of dimension $N = 1, 2, \ldots$ which we call V_N.

Let's focus on the 2-dimensional representation V_2 for now.

[1]Well, two, but they are almost identical

Example: the Jones polynomial

- The Jones polynomial can be defined in terms of a certain braid group representation $\mathcal{V}_2$.
- Let β be a braid on on b strands.
- We think of $\mathcal{V}_2(\beta)$ as a map $V_2^{\otimes b} \to V_2^{\otimes b}$ of tensor powers of V_2.
- To define $\mathcal{V}_2(\sigma) : V_2 \otimes V_2 \to V_2 \otimes V_2$, need a linear map satisfying the braid relation. (σ is a braid generator.)
- Explicitly $\mathcal{V}_2(\sigma)$ is a 4×4 matrix with entries in $\mathbb{C}[q, q^{-1}]$.

Example: the Jones polynomial

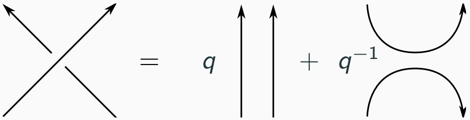
Key idea

The braiding $\mathcal{F}(\sigma)$ is given by the action of the *universal R-matrix*[2] $\mathbf{R} \in \mathcal{U}_q \otimes \mathcal{U}_q$:

$$\mathcal{F}(\sigma)(x) = \tau(\mathbf{R} \cdot x)$$

where $\tau(v \otimes w) = w \otimes v$.

Can explicitly compute the action of $\mathbf{R}$. Leads to *skein relation*



which can be used to define the Jones polynomial without using quantum groups at all.

[2]Actually it's in a sort of completion of $\mathcal{U}_q \otimes \mathcal{U}_q$. This will come up later.

Computing the Jones polynomial

To compute the Jones polynomial $\mathrm{V}_2(L)$ of a link L :

- Represent L as the closure of a braid β on b strands
- Compute the $2^b \times 2^b$ matrix $\mathcal{V}_2(\beta)$
- Its (quantum) trace is a Laurent polynomial $\mathrm{V}_2(L)$ in q^2
- This is an invariant[3] of L called the Jones polynomial

This is an example of the *Reshetikhin-Turaev construction*.

[3] Modulo some technicalities about framings that are not important here.

Algebra $\to$ topology

This process was very algebraic. I used words like:

- quantum group (a q-analog of a Lie algebra/group)
- trace
- representation (of a group/algebra)

I did *not* use more topological/geometric ideas like

- homology/fundamental groups
- essential surfaces
- geometrization

However, all this algebra still knows about geometry!

The colored Jones polynomial

- We can repeat the Resethikin-Turaev construction defining $\mathrm{V}_2(L)$ with any representation of $\mathcal{U}_q$ (or of any quantum group.)

Definition

The quantum invariant assigned to a link L by the N-dimensional irrep V_N of $\mathcal{U}_q$ is the *Nth colored Jones polynomial* $\mathrm{V}_N(L)$.

- We can do this diagrammatically in terms of cables of links, or by using *Jones-Wenzl projectors*

Value at roots of unity

We are most interested in particular values for knots K.

Set $\xi = \exp(\pi i/N)$ and normalize so that $\mathrm{V}_N(\text{unknot}) = 1$.

Definition

The complex number

$$\mathrm{J}_N(K) = \mathrm{V}_N(K)|_{q=\xi}$$

is called the *Nth quantum dilogarithm* of K.

Why the name? We will explain later.

Value at roots of unity

Figure-eight knot

Set $\{k\} = \xi^k - \xi^{-k}$. Then

$$\mathrm{J}_N(4_1) = \sum_{j=0}^{N-1} \prod_{k=1}^{j} \{N-k\}\{N+k\}.$$

- Computing closed formulas like this is hard!
- If K is presented as the closure of a braid on b strands, then computing $\mathrm{J}_N(K)$ involves the trace of a $N^b \times N^b$ matrix.

So far, only algebra

- The quantum dilogarithm (and things like it) are algebraic: coming from representation theory.
- What does it *mean* that $J_N(4_1) = \sum_{j=0}^{N-1} \prod_{k=1}^{j} \{N-k\}\{N+k\}$?

Geometric connections

Theorem

$$2\pi \lim_{N\to\infty} \frac{\log|\mathrm{J}_N(4_1)|}{N} = 2.02988\ldots = \mathrm{Vol}(4_1)$$

where $\mathrm{Vol}(K)$ *is the volume of the complete hyperbolic structure of* $S^3 \setminus K$*.*

Conjecture (Volume conjecture [Kas97; MM01])

For any hyperbolic knot K*,*

$$2\pi \lim_{N\to\infty} \frac{\log|\mathrm{J}_N(K)|}{N} = \mathrm{Vol}(K).$$

- There are versions for complex volume, for knots in 3-manifolds, for 3-manifolds...
- In every case where the left-hand limit is known to exist the conjecture holds.

How does J_N know about hyperbolic geometry?

How does J_N know about hyperbolic geometry?

- It's a conjecture, so no one really knows.
- I can now get to the main point of my talk: a program aimed at answering this sort of question.
- Along the way I hope we can define some new, even better knot invariants.

Holonomy invariants

The idea

- To describe geometry of a topological space X, pick a (conjugacy class of) representations $\pi_1(X) \to G$ for G a Lie group
- For example, a hyperbolic structure on a 3-manifold X is given by a

$$\rho : \pi_1(X) \to \mathrm{Isom}(\mathbb{H}^3) = \mathrm{PSL}_2(\mathbb{C})$$

  usually called the *holonomy representation*.
- We focus on $X = S^3 \setminus K$ a knot complement and $G = \mathrm{SL}_2(\mathbb{C})$.
- Sometimes (especially in physics contexts) we view this data as a flat $\mathfrak{sl}_2$-connection on X.

The idea

Definition

A $\mathrm{SL}_2(\mathbb{C})$-*holonomy invariant* of knots gives a scalar $F_K(\rho) \in \mathbb{C}$, where $\rho : \pi_1(S^3 \setminus K) \to \mathrm{SL}_2(\mathbb{C})$. It should depend only on the conjugacy class (gauge class) of ρ.

From now on, we say *holonomy invariant* and assume $G = \mathrm{SL}_2(\mathbb{C})$.

Another perspective

A holonomy invariant assigns a function $F_K : \mathfrak{X}(K) \to \mathbb{C}$ to every knot, where $\mathfrak{X}(K)$ is the $\mathrm{SL}_2(\mathbb{C})$-character variety of K.

Let's explain that in more detail:

The representation variety

Definition

The $SL_2(\mathbb{C})$-*representation variety* of a knot K is the space $\mathfrak{R}(K)$ of homomorphisms

$$\rho : \pi_K \to SL_2(\mathbb{C}).$$

where $\pi_K = \pi_1(S^3 \setminus K)$ is the fundamental group of the knot complement.

- It is an algebraic variety (a set cut out of $\mathbb{C}^n$ by polynomial equations)
- Two representations ρ_1, ρ_2 are *conjugate* or *gauge-equivalent* if

$$\rho_1(y) = g\rho_2(y)g^{-1}$$

for all $y \in \pi_K$ and some $g \in SL_2(\mathbb{C})$.

The character variety

We want to say that conjugate representations are the same:

Definition (Morally correct definition)

The $\mathrm{SL}_2(\mathbb{C})$-*character variety* of a knot K is

$$\mathfrak{X}(K) = \mathfrak{R}(K)/\text{conjugation}$$

- $\mathfrak{X}(K)$ captures lots of important geometric and topological information about K.
- One reason: $\mathrm{PSL}_2(\mathbb{C}) = \mathrm{SL}_2(\mathbb{C})/\{\pm 1\}$ is the isometry group of hyperbolic 3-space.
- A hyperbolic knot K comes with (two) distinguished point(s) of $\mathfrak{X}(K)$ corresponding to the finite-volume hyperbolic structure.

A technical point

- Taking a naive quotient of $\mathfrak{R}(K)$ gives a badly-behaved space (not separable, etc.) Can fix by setting

$$\begin{aligned} \mathfrak{X}(K) &= \text{Spec}(\text{conjugation-invariant functions on } \mathfrak{R}(K)) \\ &= \text{Spec}(\text{characters of } \mathrm{SL}_2(\mathbb{C}) \text{ reps of } \pi_K\ ) \\ &= \text{Spec}(\text{algebra generated by trace functions } \mathrm{tr}_x : \rho \mapsto \mathrm{tr}\,\rho(x)) \end{aligned}$$

  hence the name.
- When we do this we throw out indecomposable but reducible ρ. Not usually a big deal: ρ with irreducible image are the most important geometrically.
- If ρ has completely reducible image it factors through the abelianization $H_1(S^3 \setminus K)$ of π_K: good simple examples.

Takeaway on character varieties

A function $f : \mathfrak{X}(K) \to \mathbb{C}$ is simply a conjugation-invariant function

$$f : \mathfrak{R}(K) \to \mathbb{C}$$

Holonomy invariants

Definition

A $\mathrm{SL}_2(\mathbb{C})$-*holonomy invariant* F assigns every knot a function

$$F_K : \mathfrak{X}(K) \to \mathbb{C}.$$

- Equivalently, a holonomy invariant is a function on pairs $(K, \rho : \pi_K \to \mathrm{SL}_2(\mathbb{C}))$ that depends only on the conjugacy class of ρ.
- Some of our examples use a slight variant (a N-fold cover) of $\mathfrak{X}(K)$
- We can think of $\rho \in \mathfrak{X}(K)$ as the holonomy of a flat $\mathfrak{sl}_2$-connection, hence the name.

Examples of holonomy invariants

Torsion

The Reidemeister torsion $\tau(K, \rho) = \tau(S^3 \setminus K, \rho)$ depends on K and $\rho \in \mathfrak{R}(K)$. It is gauge-invariant, so we get a function

$$\tau(K, -) : \mathfrak{X}(K) \to \mathbb{C}$$

i.e. a holonomy invariant.

Examples of holonomy invariants

Complex volume

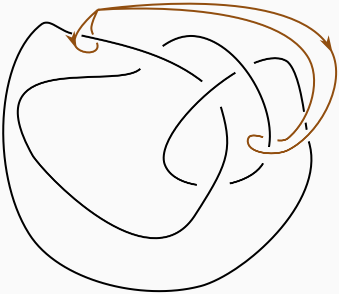
The *complex volume* of a hyperbolic knot

$$\mathrm{Vol}(K) + i\,\mathrm{CS}(K) \in \mathbb{C}/i\pi^2\mathbb{Z}$$

can be computed by evaluating a certain characteristic class of flat $\mathrm{PSL}_2(\mathbb{C})$-bundles on the finite-volume hyperbolic structure of $S^3 \setminus K$. We can think of this as a holonomy invariant by evaluating that class on other elements of $\mathfrak{X}(K)$.

The knot group

- K a knot in S^3
- $\pi_K = \pi_1(S^3 \setminus K)$ is finitely generated, say by *meridians*
- All meridians of K are conjugate



Two meridians of the figure-eight knot

Taking the abelian limit

- Before, we mentioned that trivial/abelian reps ρ give ordinary knot invariants.
- Consider the one-dimensional family of representations defined on meridians by

$$\alpha_t(x) = \begin{bmatrix} t & 0 \\ 0 & t^{-1} \end{bmatrix}$$

- The α_t have abelian image and exist for any knot K.

Theorem

$$\tau(K, \alpha_t) = \nabla_K(t)\nabla_K(t^{-1}) \\ = \nabla_K(t)^2$$

where ∇_K is the Conway potential of K (up to a constant, the Alexander polynomial).

Taking the abelian limit

Proof.

α_t is reducible, so $\tau(K, \alpha_t)$ factors into two pieces. Each factor is an abelianization map on group rings $\mathbb{Z}[\pi_K] \to \mathbb{Z}[\mathbb{Z}] = \mathbb{Z}[t, t^{-1}]$. By the usual arguments, this is the abelian Reidemeister torsion/Alexander polynomial. □

Takeaway

In general $\mathfrak{X}(K)$ is complicated and depends on K, but for *any* K $\alpha_t \in \mathfrak{X}(K)$. We can evaluate any holonomy invariant on α_t to get an easier example.

Examples of quantum holonomy invariants

Some algebra

- Key algebra ingredient: representation theory of $\mathcal{U}_\xi$ for $q = \xi = \exp(\pi i/N)$ or $q = \zeta = \exp(2\pi i/N)$ a root of unity
- The center of $\mathcal{U}_\xi$ is birationally equivalent to (a finite cover of) $\mathrm{SL}_2(\mathbb{C})$
- In particular there is a family of simple $\mathcal{U}_\xi$-modules indexed by $\mathrm{SL}_2(\mathbb{C})$
- Roughly speaking we assign a strand with holonomy $g \in \mathrm{SL}_2(\mathbb{C})$ a module with central character g
- More details after some examples

The Kashaev–Reshetikhin invariant

Extending Kashaev and Reshetikhin [KR05], myself, Chen, Morrison, and Snyder [Che+21] constructed a holonomy invariant.

Fact

For $g \in \mathrm{SL}_2(\mathbb{C})$, write χ_g for the associated central $\mathcal{U}_\zeta$-character and $\ker \chi_g$ the ideal generated by its kernel. $\mathcal{U}_\zeta / \ker \chi_g$ is a simple bimodule of dimension N^2.

(This is not quite right: details later!)

Theorem

By assigning a strand of a knot diagram with holonomy g the module $\mathcal{U}_\zeta / \ker \chi_g$, we get a holonomy invariant $\mathrm{KR}(K, \rho)$ of knots. $\mathrm{KR}(K, -)$ is a rational function on a N-fold cover $\mathfrak{X}_N(K)$ of $\mathfrak{X}(K)$.

Extended character variety

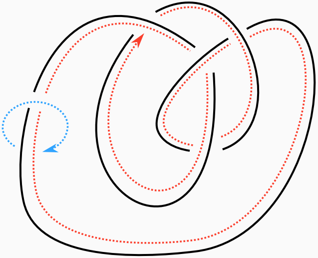
- Recall that any two meridians of a knot K are conjugate (if they match orientation)
- For $\rho \in \mathfrak{X}(K)$, eigenvalues of $\rho(x)$ for x a meridian are independent of x
- A *fractional eigenvalue* of ρ is a μ with $\operatorname{tr} \rho(x) = \mu^N + \mu^{-N}$

Definition

Points of $\mathfrak{X}_N(K)$ are pairs (ρ, μ) with $\rho \in \mathfrak{X}(K)$ and μ a fractional eigenvalue of ρ.

Immediate that $\mathfrak{X}_N(K) \to \mathfrak{X}(K)$ is an N-fold cover.

The figure-eight knot



$K = 4_1$

longitude meridian

$$\mathfrak{X}(4_1) = \mathbb{C}[M^{\pm 1}, L^{\pm 1}] / \langle (L-1)(L^2 M^4 + L(-M^8 + M^6 + 2M^4 + M^2 - 1) + M^4) \rangle$$

$M^{\pm 1}$ are the eigenvalues of the meridian and $L^{\pm 1}$ are the eigenvalues of the longitude.

To get $\mathfrak{X}_N(4_1)$, replace M with $\mu^N = M$

The figure-eight knot

$(L-1)$ factor is the *commutative* component and the other is *geometric*. We compute that, for $N=3$,

$$\mathrm{KR}(K)_{\mathsf{comm}} = \left(\mu^4 + 3\mu^2 + 5 + 3\mu^{-2} + \mu^{-4}\right)^2$$
$$\mathrm{KR}(K)_{\mathsf{geom}} = 3(\mu^2 + \mu^{-2})(\mu + 1 + \mu^{-1})^3(\mu - 1 + \mu^{-1})^3$$

Complete hyperbolic structure of 4_1 complement corresponds to points $\mu = 1, \exp(2\pi i/3), \exp(4\pi i/3)$ on geometric component.

Observation

$\mathrm{KR}(K)_{\mathsf{geom}}$ vanishes for μ a primitive root of unity. Seems to extend to other knots and odd N for $\zeta = \exp(2\pi i/N)$; does not occur for $\xi = \exp(\pi i/N)$ and N even.

A-polynomial curve

- $\mathfrak{X}(4_1)$ had two components parametrized by M and L
- In general, $\mathfrak{X}(K)$ could have more components and might not be canonically parametrized by M and L
- Commutative component is always parametrized by M
- Restriction to just M and L gives the *A-polynomial curve*

Theorem (Dunfield [Dun99])

For hyperbolic K, geometric component of $\mathfrak{X}(K)$ is canonically parametrized by M and L.

Corollary

For hyperbolic K, $\mathrm{KR}(K)$ is a rational function on the commutative and geometric components of the A-polynomial curve.

What about the Jones polynomial?

In the abstract I promised a holonomy invariant extending the colored Jones polynomial. More precisely:

Goal

A holonomy invariant J_N such that $\mathrm{J}_N(K, \pm 1)$ (the value at the trivial representation) recovers the quantum dilogarithm $\mathrm{J}_N(K)$ (colored Jones at a root of unity).

It extends the quantum dilogarithm to $\rho \in \mathfrak{X}_N(K)$ with nonabelian image, so we call J_N the *nonabelian quantum dilogarithm*.

Why do we care?

- The modules V_N for $q = \exp(\pi i/N)$ are associated to the holonomy sending every meridian to $(-1)^{N+1}$.
- Thus, the volume conjecture is about relating $\mathrm{J}_N(K, \alpha_{(-1)^{N+1}})$ and hyperbolic volume.
- The value $\mathrm{J}_N(K, \rho_{\mathsf{hyp}})$ at the complete finite-volume hyperbolic structure should know about the volume
- We might be able to relate their asymptotics via ideas like:
    - analytic continuation,
    - resurgence,
    - AJ conjecture,
    - others?

Defining J_N

Fact

For each (generic) $g \in \mathrm{SL}_2(\mathbb{C})$ there are N irreducible $\mathcal{U}_\xi$-modules $V_{g,\mu}$ parametrized by fractional eigenvalues of g. For $g = (-1)^{N+1}$ and $\mu = \xi^{N-1}$ we recover the module V_N defining the colored Jones polynomial.

Theorem (part of my PhD thesis)

There is a holonomy invariant J_N assigning $V_{g,\mu}$ to strands. It is well-defined up to a $2N$th root of unity.

This is an extension of work of Blanchet, Geer, Patureau-Mirand, and Reshetikhin [Bla+20]. My improvements are:

- Explicitly computing the braiding matrices
- Working out a more geometric way to describe ρ for link complements

The abelian case

Kashaev's quantum dilogarithm

When $\rho = \alpha_{(-1)^{N+1}}$ is $\pm$ the trivial representation,

$$\mathrm{J}_N(K, \alpha_{(-1)^{N+1}}) = J_N(K)$$

is the quantum dilogarithm, i.e. the Nth colored Jones polynomial evaluated at $\exp(2\pi i/N)$.

The Akutsu-Deguchi-Ohtsuki invariant

When $\rho = \alpha_t$ and $t \neq \pm 1$,

$$\mathrm{J}_N(K, \alpha_t) = \mathrm{ADO}_N(t)$$

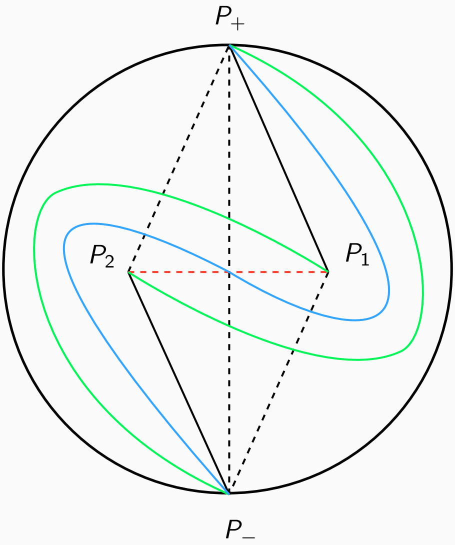
is the Nth ADO invariant.

The ADO invariant is a higher-order Alexander polynomial. When $N = 2$, it is exactly the Conway potential/Alexander polynomial/abelian Reidemeister torsion.

Shaped link diagrams

- In this coordinate system we assign *shapes*:
    - a complex number b_i to each segment of a link diagram
    - a meridian eigenvalue m_i to each link component
- Gives $\rho : \pi_1(S^3 \setminus L) \to \mathrm{SL}_2(\mathbb{C})$ when they satisfy a certain set of equations
- These are *exactly* the octahedral gluing equations of Kim, Kim, and Yoon [KKY18].
- Related to an ideal triangulation with four ideal tetrahedra at each crossing

Octahedral decomposition



What is a dilogarithm?

- Can compute complex volume by evaluating a special function called the *Rogers dilogarithm* on the shape parameters of the tetrahedra
- Kashaev described a matrix analogue called the *quantum dilogarithm*
- Used it to define a knot invariant which turns out to be the colored Jones at a root of unity (nontrivial to show!)
- The braiding defining J_N uses four quantum dilogarithms, one for each tetrahedron

Another holonomy invariant

Quantum hyperbolic invariants

Baseilhac and Benedetti [BB04] constructed *quantum hyperbolic invariants* of 3-manifolds with links inside them via state-sums and triangulations.

- They used quantum dilogarithms, just like in our construction
- Their invariants appear to be closely related to our nonabelian quantum dilogarithm.
- Our version is much more clearly related to the Jones polynomial
- We can also prove a relation with the torsion:

Relation with the torsion

Theorem (Me [McP21])

For any link L and $\rho \in \mathfrak{X}_2(L)$ that does not have 1 as an eigenvalue,

$$\mathrm{J}_2(L, \rho)\mathrm{J}_2(\overline{L}, \rho) = \tau(S^3 \setminus L, \rho)$$

where $\overline{L}$ is the mirror image and τ is the Redemeister torsion twisted by ρ.

Proof idea.

There is a Schur-Weyl duality between the braiding for $\mathcal{U}_\xi$ defining J_2 and the twisted Burau representation defining τ. Need to use a "quantum double" to get the norm-square on the left hand side. □

Constructing quantum holonomy invariants

How to construct them

- I will now give a very rapid overview of some of the algebra used to define these invariants
- Happy to discuss more details if you want to know them!

Holonomy Reshetikhin-Turaev

- Recall that in the RT construction of knot invariants from $\mathcal{U}_q = \mathcal{U}_q(\mathfrak{sl}_2)$, we think about putting a $\mathcal{U}_q$-module V on each strand of the knot K.
- For a pair (K, ρ), each strand has a meridian x and holonomy $\rho(x) \in \mathrm{SL}_2(\mathbb{C})$.
- We want to find a family of modules V_g parametrized by points of $\mathrm{SL}_2(\mathbb{C})$.
- Actual answer is a bit more complicated: let's explain why.

Roots of unity

Recall that for generic q, $\mathcal{U}_q$-modules are essentially indexed by a single integer (the highest weight), just like for ordinary $\mathfrak{sl}_2$.
However, for $q = \xi = \exp(\pi i/N)$ a primitive $2N$th root of 1, $\mathcal{U}_\xi$-modules are much more interesting.

Theorem

N-dimensional projective simple $\mathcal{U}_\xi$-modules are indexed by:

1. *a (generic) point $g \in \mathsf{SL}_2(\mathbb{C})$*
2. *an Nth root μ of an eigenvalue of g*

Why?

Central characters

- $\mathcal{U}_q$ has generators $E, F, K = q^H$ (like $\mathfrak{sl}_2$)
- At $q = \xi$, get central subalgebra $\mathcal{Z}_0 = \mathbb{C}[E^N, F^N, K^{\pm N}]$
- For central characters $\chi : \mathcal{Z}_0 \to \mathbb{C}$,

$$\chi \in \operatorname{Spec} \mathcal{Z}_0 \leftrightarrow \left(\begin{bmatrix} \chi(K^N) & 0 \\ \chi(K^N F^N) & 1 \end{bmatrix}, \begin{bmatrix} 1 & \chi(E^N) \\ 0 & \chi(K^N) \end{bmatrix} \right)$$

$$\leftrightarrow \begin{bmatrix} \chi(K^N) & -\chi(E^N) \\ \chi(K^N F^N) & \chi(K^N) - \chi(K^N E^N F^N) \end{bmatrix} \in \mathrm{SL}_2(\mathbb{C})$$

- Action of central Casimir Ω given by Nth root, full center is $\mathcal{Z} = \mathcal{Z}_0[\Omega]/(\text{polynomial relation})$
- Characters $\chi : \mathcal{Z} \to \mathbb{C}$ are in bijection with simple $\mathcal{U}_\xi$-modules.

Factorized groups

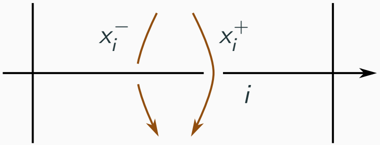
- $\mathcal{U}_\xi$-modules are really graded by the group $\mathrm{SL}_2(\mathbb{C})^*$ of pairs

$$\chi = (\chi^+, \chi^-) = \left(\begin{bmatrix} \kappa & 0 \\ \phi & 1 \end{bmatrix}, \begin{bmatrix} 1 & \epsilon \\ 0 & \kappa \end{bmatrix} \right)$$

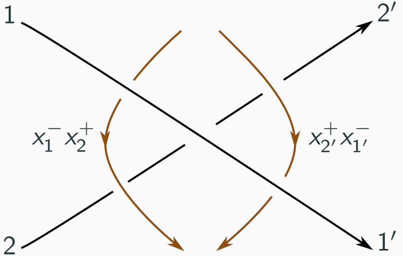
- This is birationally equivalent to $\mathrm{SL}_2(\mathbb{C})$, but not isomorphic as a group.
- Leads to slightly unusual description of π_K.

- Usual description (Wirtinger presentation) of knot group from a diagram has one generator for each arc.
- We instead want a *groupoid* with two generators for each segment.
- Path above a segment labeled by χ gives χ^+, path below gives χ^-

Fundamental groupoid



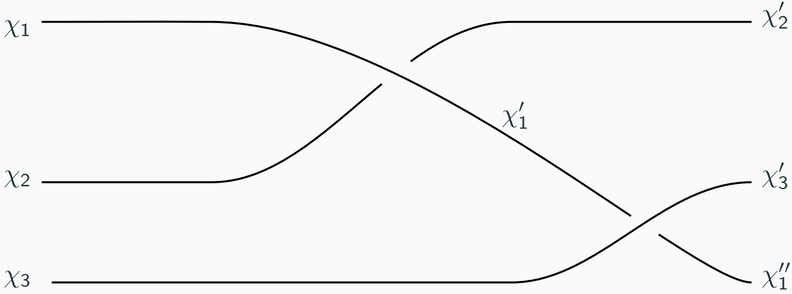
The generators associated to segment i



There are relations at each crossing, such as the above

Colored braids

To represent (K, ρ) as a braid closure, decorate segments with characters χ giving holonomies:



The braid action on the χ_i is equivalently given by the rules on the previous slide, or by the braiding on $\mathcal{U}_\xi$. It forms a *biquandle*.

The colored braid groupoid

Just like we have braid groups for braids, we can organize these into a groupoid:

Definition

The $\mathrm{SL}_2(\mathbb{C})$-*colored braid groupoid* $\mathbb{B}_N(\mathrm{SL}_2(\mathbb{C}))$ is a category:

objects tuples $(\chi_1, \dots, \chi_n)$ of characters $\chi_i : \mathcal{Z}_0 \to \mathbb{C}$

morphisms braids $\beta : (\chi_1, \dots, \chi_n) \to (\chi_1', \dots, \chi_n')$

Closures of colored braids are links L plus $\rho \in \mathfrak{X}(L)$.

Braid groups

The ordinary braid group is the component with $\chi_1 = \cdots = \chi_n = \mathrm{id}$. (Recall a groupoid with one object is a group.)

Invariants from the braid groupoid

To define a holonomy invariant, we need

1. a functor $\mathcal{F} : \mathbb{B}_N(\mathsf{SL}_2(\mathbb{C})) \to \mathcal{U}_\xi$-Mod (must satisfy colored Reidemeister moves!)
2. an trace on endomorphisms of $\mathcal{U}_\xi$-Mod

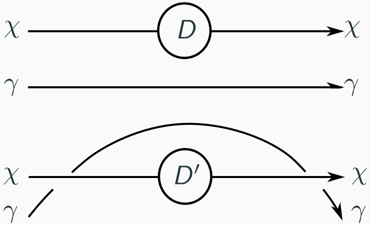
Theorem

Given such an $\mathcal{F}$, the trace of $\mathcal{F}(\beta)$ is an invariant of the closure (L, ρ) of β.

Getting the trace is trickier than normal because of non-semimplicity: need to use *modified dimensions/traces*.

Gauge invariance

It turns out that, given such a functor $\mathcal{F}$, the associated invariant is automatically gauge-invariant! Two pictures explain why:



Here D' is gauge-equivalent to D.

Questions?

Bonus: Why is it called a quantum dilogarithm?

The dilogarithm

- The dilogarithm is

$$L_2(x) = -\int_0^x \frac{\log(1-z)}{z}\, dz$$

  and Rogers' dilogarithm is

$$L(x) = L_2(x) + \log(1-x)\log(x)/2.$$

  $L(x)$ can be used to compute complex volumes of tetrahedra, hence of manifolds.

- It satisfies the 5-term relation

$$L(x) + L(y) - L(xy) = L\left(\frac{x-xy}{1-xy}\right) + L\left(\frac{y-xy}{1-xy}\right)$$

  which is related to the 3-2 move on triangulations

The quantum dilogarithm

- Faddeev and Kashaev showed the q-series

$$\Psi(x) = \prod_{n=1}^{\infty}(1 - xq^n)$$

  is a q-analog of $L(x)$ and satisfies a noncommutative 5-term relation.

- The *cyclic quantum dilogarithm*

$$L(B, A|n) = \prod_{k=1}^{n}(1 - \xi^{2k}B)/A$$

  for $A^N + B^N = 1$ is a root-of-unity analogue of $\Psi(x)$.

Link invariants from the quantum dilogarithm

- By taking a certain singular limit Kashaev defined his quantum dilogarithm invariant.
- By replacing Rogers dilogarithms $L(x)$ with cyclic dilogarithms $L(B, A|n)$, Baseilhac and Benedetti defined holonomy invariants B_N for triangulated 3-manifolds with links inside them.
- B_N is constructed as a state-sum, with one function $L(B, A|n)$ for each tetrahedron.

The nonabelian quantum dilogarithm

- Even though the definition of J_N appears quite different from B_N, recent computations of the braiding show they are closely related.
- In particular, the braiding defined by $\mathcal{J}_N$ factors into a product of four linear maps, each of which is associated to a tetrahedron in the octahedral decomposition of the knot complement.
- To emphasize the connection with Kashaev's construction and the incorporation of *nonabelian* $\rho \in \mathfrak{X}_N(K)$, we used the name *nonabelian quantum dilogarithm*.

The nonabelian quantum dilogarithm and the torsion

An explicit relationship

Theorem (C. [McP21])

For any $\rho \in \mathfrak{X}_2(K)$,

$$\mathrm{J}_2(K, \rho)\mathrm{J}_2(\overline{K}, \rho) = \tau(K, \rho)$$

where $\overline{K}$ *is the mirror image of* K.

Comparing

$$\nabla_K(t)\nabla_{\overline{K}}(t) = \tau(K, \alpha_t)$$

we think of $\mathrm{J}_2(K, \rho)$ as a *nonabelian Conway potential*.

How do we compute the right-hand side? Use the Burau representation.

The Burau representation

Consider colored braids on b strands. Write $\rho = (\chi_1, \ldots, \chi_b)$ for an object of $\mathbb{B}_2(\mathsf{SL}_2(\mathbb{C}))$, equivalently a representation

$$\rho : \pi_1(D_b) \to \mathsf{SL}_2(\mathbb{C})$$

where D_b is a b-punctured disc. Let β be a braid on b stands, i.e. an element of $\mathsf{Map}(D_b, \partial D_b)$. As a colored braid, it becomes a morphism $\beta : \rho \to \rho'$.

Definition

The *Burau representation* is the action on twisted locally-finite homology:

$$\mathcal{B}(\beta) : \mathrm{H}_1(D_b; \rho) \to \mathrm{H}_1(D_b; \rho')$$

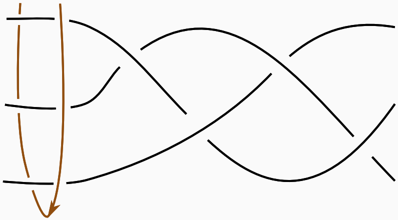
induced by the action of β on D_b.

Computing the torsion

Proposition

If (K, ρ) is the closure of β, then

$$\tau(K,\rho) = \frac{\det(1 - \mathcal{B}(\beta))}{\det(1 - \rho(y))}$$



y is a path around every strand, as above.

Determinant to trace

To make this a trace, let $\bigwedge \mathcal{B}$ be the action on the exterior algebra of homology. Then

$$\operatorname{str}\left(\bigwedge \mathcal{B}(\beta)\right) = \det(1 - \mathcal{B}(\beta)).$$

Here str is the $\mathbb{Z}/2$-graded trace: multiply action on degree k part by $(-1)^k$.

Multiplicity spaces

We want to understand $\mathcal{J}_2(\beta) : \mathcal{J}_2(\rho) \to \mathcal{J}_2(\rho)$, $\rho = (\chi_1, \ldots, \chi_b)$. First we need to understand $\mathcal{J}_2(\rho)$. Use:

Proposition

$$\mathcal{J}_2(\rho) = \bigotimes_{i=1}^{b} V_{\chi_i} \cong X^+ \otimes_{\mathbb{C}} V_{\chi_+} \oplus X^- \otimes_{\mathbb{C}} V_{\chi_-}$$

Here:

- $\chi_\pm$ are characters corresponding to the total holonomy $\rho(y)$
- there are two because there are two choices $\pm\mu$ of fractional eigenvalue for $\rho(y)$
- Action of $\mathcal{J}_2(\beta)$ factors through *multiplicity spaces* $X^\pm$

Schur-Weyl duality

Theorem (Me [McP21])

There is a subalgebra $\mathfrak{C}_b$ of $\mathcal{U}_\xi^{\otimes b}$ that

1. *(super)commutes with the image of $\Delta\mathcal{U}$ in the tensor power,*
2. *is isomorphic as a vector space to $\bigwedge \mathcal{B}(\chi_1, \dots, \chi_b)$,*
3. *such that the braid group action on $\mathfrak{C}_b \subseteq \mathcal{U}_\xi^{\otimes b}$ agrees with $\mathcal{B}$.*

Compare Schur-Weyl duality between tensor powers of SL_n and the symmetric group.

Computing J_2

Corollary (Wrong)

The $\mathbb{Z}/2$-graded multiplicity space $X = X^+ \oplus X^-$ is isomorphic to $\bigwedge \mathcal{B}(\rho)$. This is compatible with the braid action, so $\mathcal{J}_2(\beta)$ acts on X by $\bigwedge \mathcal{B}(\beta)$.

The theorem about $\tau(K, \rho)$ would follow immediately, except that this is false!

Fixing the idea

- The problem is that $\mathfrak{C}_b$ does not act faithfully on $\mathcal{J}_2(\chi_1, \ldots, \chi_n)$.
- Among other reasons, dimensions don't match.
- To fix, consider a "quantum double"

$$\mathcal{T}_2 = \mathcal{J}_2 \boxtimes \overline{\mathcal{J}}_2$$

- Then the theorem works and

$$\begin{aligned} \tau(K, \rho) &= \mathrm{T}_2(K, \rho) && \text{(by Schur-Weyl)} \\ &= \mathrm{J}_2(K, \rho)\mathrm{J}_2(\overline{K}, \rho) && \text{(by definition)} \end{aligned}$$

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