(Towards?) A quantization of the $SL_2(\mathbb{C})$ Chern-Simons invariant

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Plan:

- quantum $SL_2(\mathbb{C})$ Chern Simons theory means ???
- Reminder on quantum invariants
  - Reshetikhin–Turaev construction
  - Kashaev invariant
- Tangles with G-structures

- Classical $SL_2(\mathbb{C})$ Chern Simons
- New invariant J_N
  - sketch definition
  - sketch construction
  - known special cases

Physics background

(maybe wrong! motivation,
not precise)

(Quantum) Chern-Simons

M = 3-mfld $A \in \Omega^1(M; \mathfrak{g})$

$$S(A) = (\text{const}) \int_M \operatorname{tr} A \wedge dA + \frac{2}{3} \operatorname{tr} A \wedge A \wedge A$$

Chern-Simons functional

defined mod $\mathbb{Z}$

Witten considers

$$Z(M) = \int_{A \in \Omega^1(M; \mathfrak{g})} e^{2\pi i k S(A)}$$

maybe do gauge-fixing first

$k \in \mathbb{Z}$ level

- For $\mathfrak{g}$ a real compact Lie alg we get Witten-Reshetikhin-Turaev Invs

- Can define using quantum groups.

What if $\mathfrak{g} = sl_2(\mathbb{C})$ is complex?

Now moduli space $\mathcal{X}(M)$ of flat connections is much bigger.

$$Z(M) = \int_{A \in \Omega^1(M, sl_2)} e^{2\pi i k S(A)}$$

too big!

decompose over "background" ρ as

$$Z(M) = \int_{\rho \in \mathcal{X}(M)} Z(M, \rho)$$

noncompact complex manifold

Each piece $Z(M,\rho)$ is an invariant of M <u>plus</u>

$$\rho : \pi_1(M) \longrightarrow SL_2(\mathbb{C})$$

ρ = holonomy of background gauge field

We seek a mathematical definition of $Z(M,\rho)$

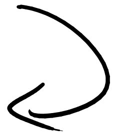
Math starts here

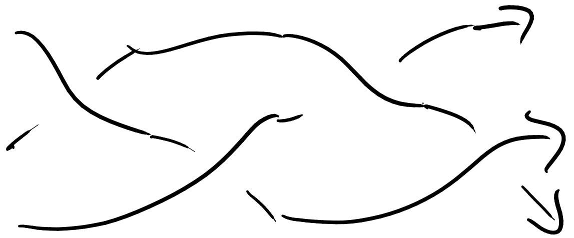
Reshetikhin–Turaev construction

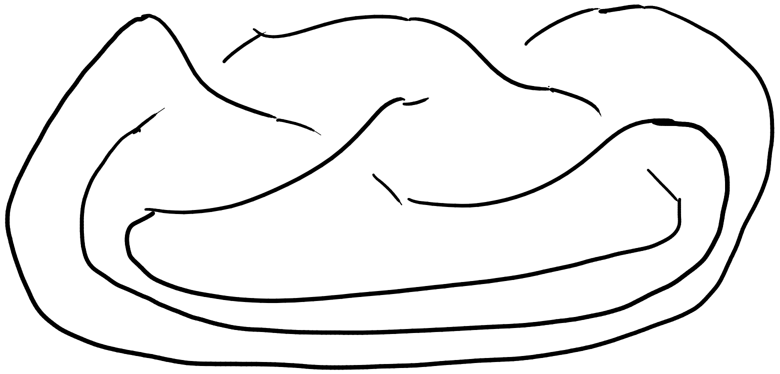
V = module over nice Hopf algebra (quantum group)

 $= \mathrm{id}_V$

 = braiding $V \otimes V \to V \otimes V$

 = evaluation $V \otimes V^* \to \mathbb{C}$

 = a tensor $V^{\otimes 3} \to V^{\otimes 3}$ whose trace is an invariant of

 = figure-eight knot

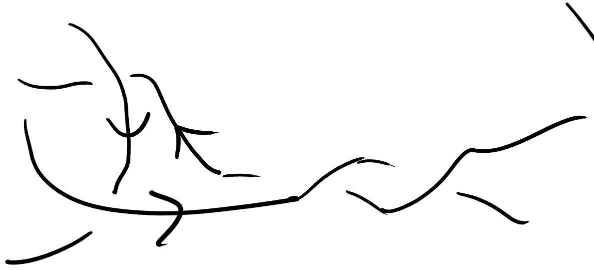
If $V =$ N-dim'l irrep of $\mathcal{U}_q(\mathfrak{sl}_2)$
(highest wt $\frac{N-1}{2}$)

get framed oriented knot (and link) invariants

$J_N(K, q)$ Nth colored Jones polynomial
(Kirillov–Reshetikhin invariant)

Laurent polynomial in q

Also have tangle invs:

$J_N\Big($  $\Big) =$

$\mathcal{U}_q(\mathfrak{sl}_2)$-module intertwiner

$V \otimes V^* \otimes V \otimes V^* \to V \otimes V^*$

depends only on framed isotopy class

If $q = e^{\pi i/N}$ get Kashaev invariant

$$J_N(K) \in \mathbb{C}$$

Kashaev had very different definition: nontrivial [Murakami-Murakami] to show equivalence

His definition assigns quantum dilog to each 3-simplex in a triangulation

Ours does to, but with nontrivial geometric params.

Goal: For $N \geq 2$, define a function

$$J_N : \mathcal{X}(S^3 \setminus K) \longrightarrow \mathbb{C}$$

character variety

$$\mathcal{X}(M) = \frac{\mathrm{Hom}(\pi_1(M), SL_2(\mathbb{C}))}{\text{conjugation}} = \frac{\text{flat connections}}{\text{gauge}}$$

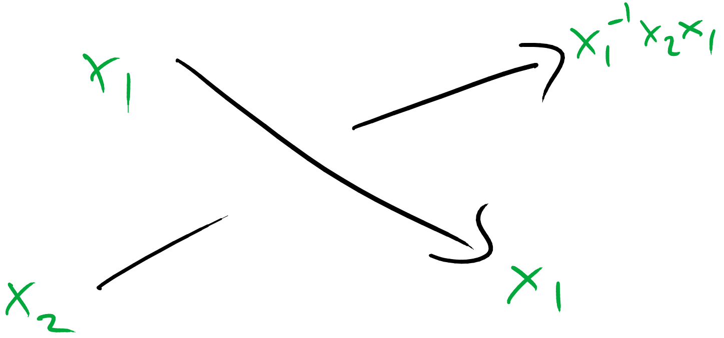
so that

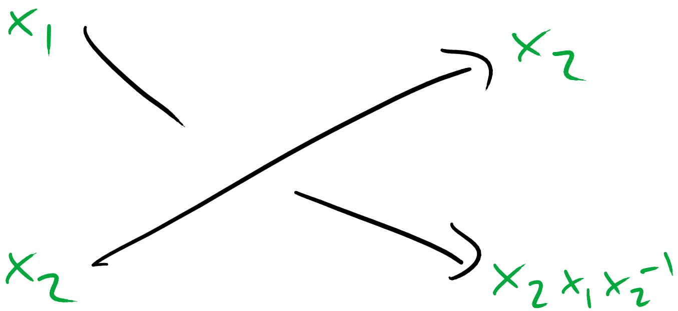
- $J_N(K, 1) =$ Kashaev inv.
- $J_N(K, \rho)$ depends nontrivially on ρ

Encode $\rho : \pi(S^3 \setminus K) \to SL_2(\mathbb{C})$ using Wirtinger presentation.

- one generator for each arc

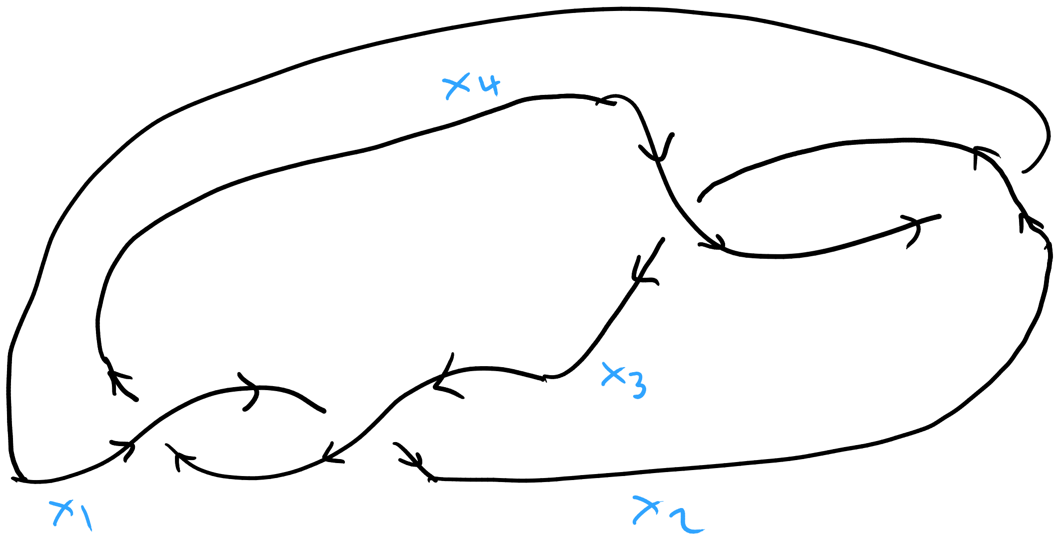
- relations



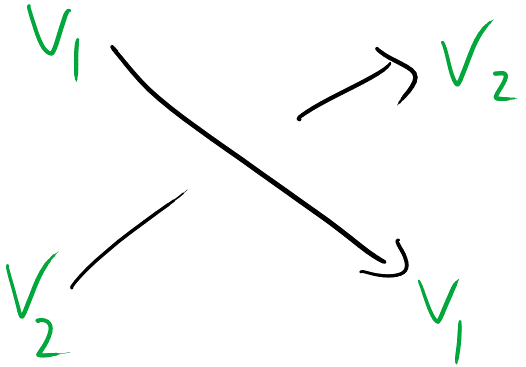


Ex: Figure-eight group is

$$\left\langle x_1, x_2, x_3, x_4 \;\middle|\; \begin{array}{l} x_4 = x_1^{-1} x_3 x_1 \\ x_2 = x_3^{-1} x_1 x_3 \\ x_2 = x_4^{-1} x_3 x_4 \\ x_4 = x_2^{-1} x_1 x_2 \end{array} \right\rangle$$



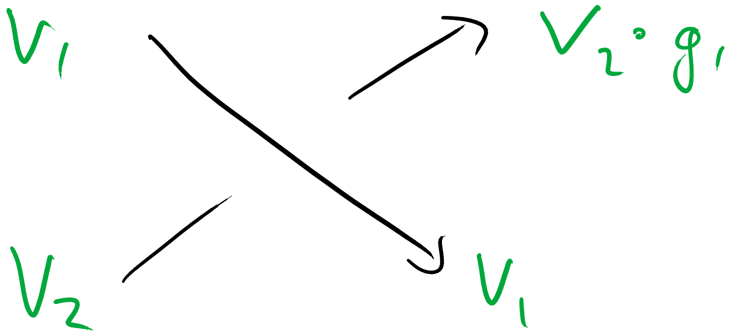
Quantum invariants



$$V_1 \otimes V_2 \longrightarrow V_2 \otimes V_1$$

braiding

G-graded quantum invariants



$$V_1 \otimes V_2 \longrightarrow (V_2 \cdot g_1) \otimes V_1$$

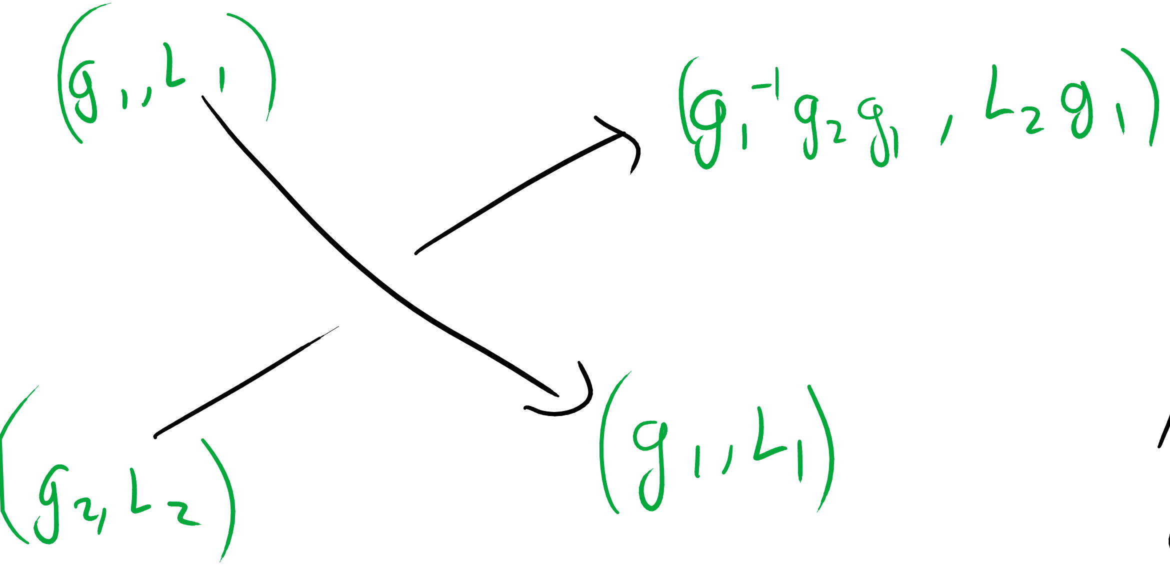
G-crossed braiding

grading $|V_1| = g_1$, $|V_2| = g_2$

G-action $|V_2 \cdot g_1| = g_1^{-1} g_2 g_1$

Generalization 1!

Add decorations to Wirtinger generators

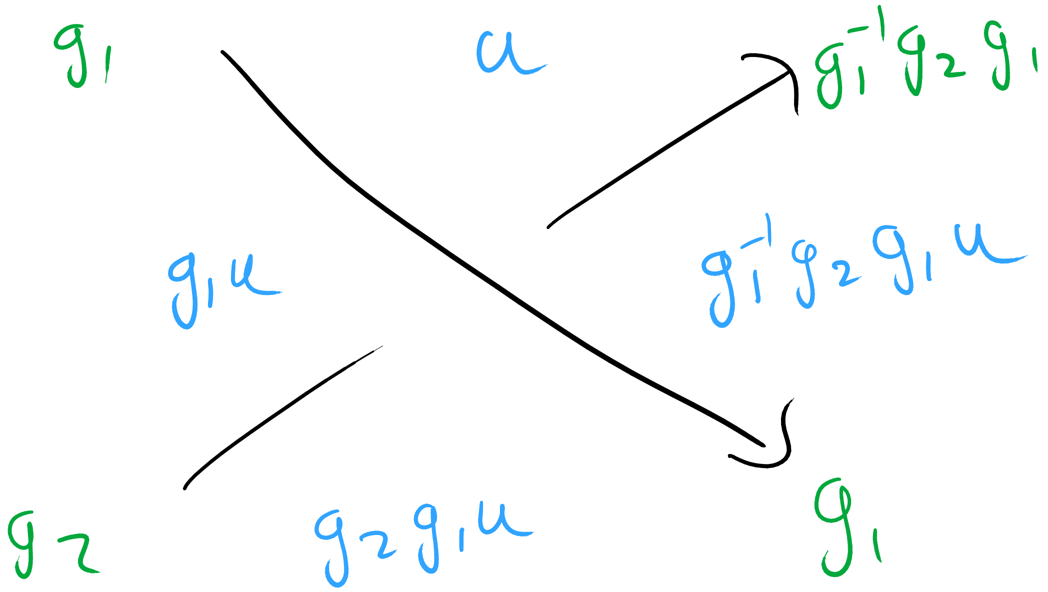


decoration = right eigenline for g

Naturally appear in hyperbolic geometry and Chern-Simons theory

Generalization 2:

Add shadows to regions subject to



shadow = column vector in $\mathbb{C}^2$

(more generally elt. of a left G-module)

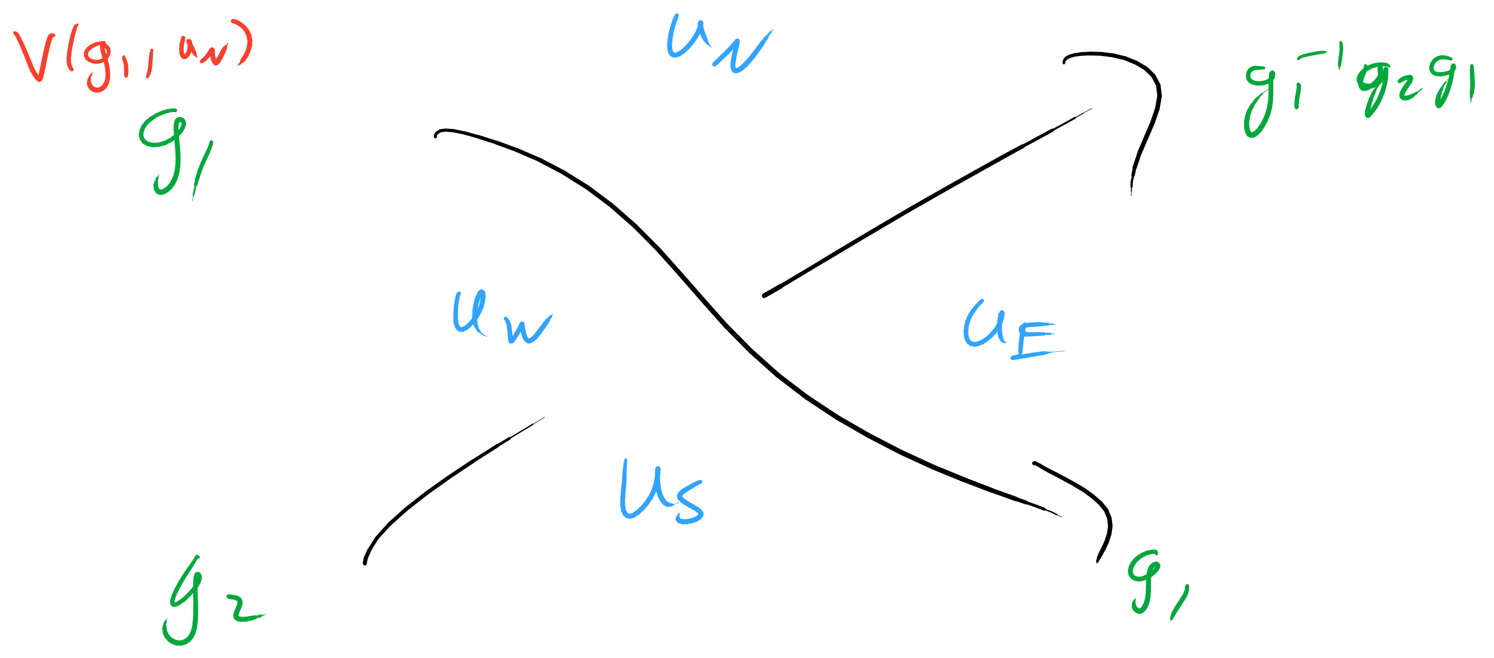
Think of choice of u_j as gauge-fixing.

Strategy: Choose (geometric version of Reshetikhin–Turaev)

- family of vector spaces $V(g, u)$ $\qquad \mathfrak{g} \xrightarrow{u} \mathcal{U}_q(\mathfrak{sl}_2)$ modules

- family of tensors to $SL_2(\mathbb{C})$ shadow-colored tangles obeying Reidemeister moves $\qquad \mathcal{U}_q(\mathfrak{sl}_2)$ intertwiners

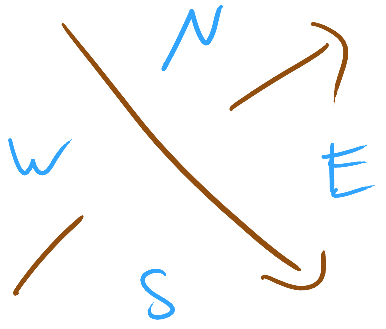
$\rightsquigarrow$ geometric tangle, link invariants

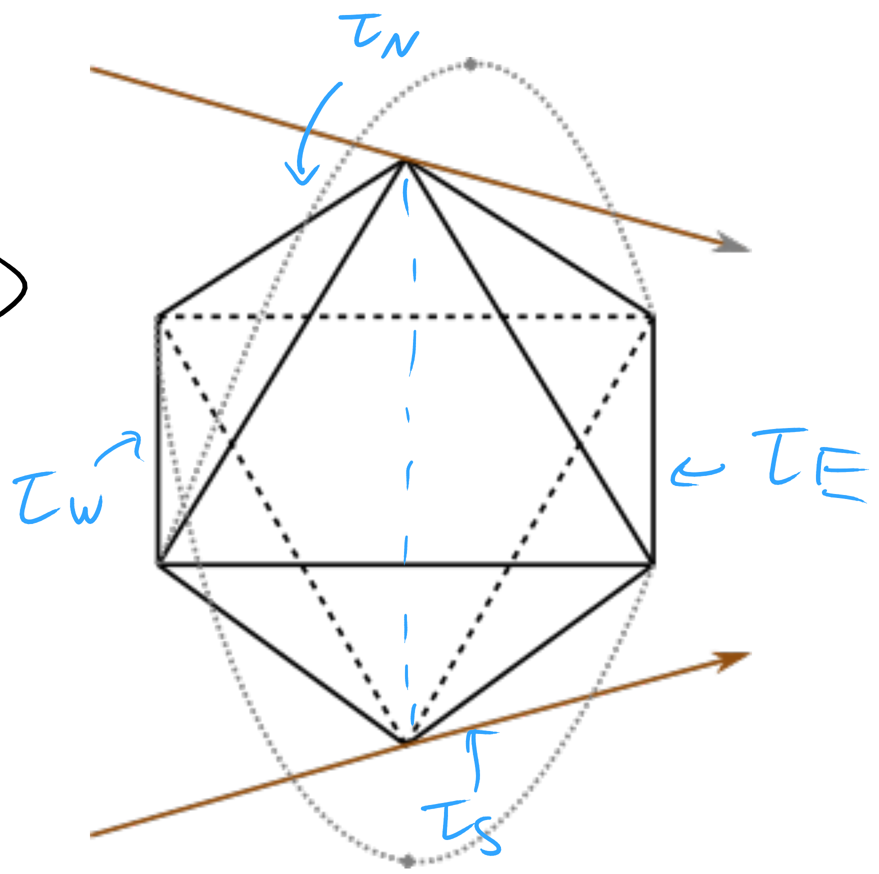


$V(g_1, u_N)$
g_1
u_N
$g_1^{-1} g_2 g_1$
u_W
u_E
u_S
g_2
g_1

Example : Hyperbolic volume

- Set $V(g,u) = \mathbb{C}$ for all g, u (Invertible theory!)
- Crossing has 4 ideal tetrahedra in octahedral decomposition





$SL_2(\mathbb{C}) \longrightarrow PSL_2(\mathbb{C}) = \mathrm{Isom}(\mathbb{H}^3)$, structure determines volume.

Concretely for τ tetrahedron

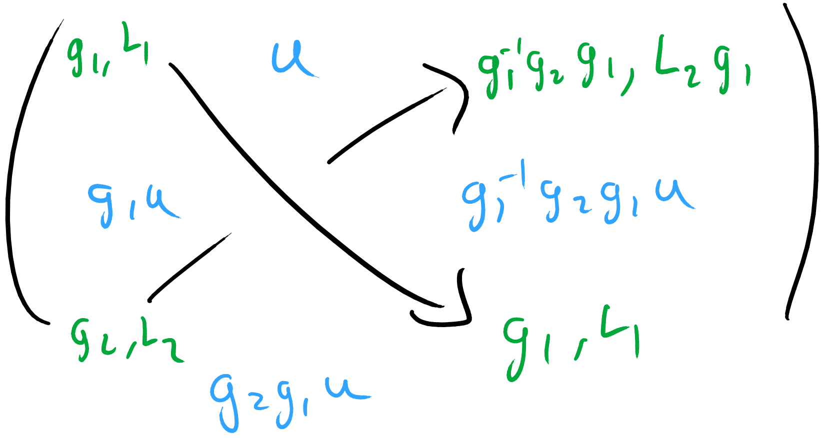
Bloch-Wigner dilogarithm

$$\mathrm{Vol}(\tau) = D(z) = \mathrm{Im}\, Li_2(z) + \log|z| \arg(1-z)$$

shape parameter = cross-ratio of vertices in $\mathbb{CP}^1$

$$Li_2(z) = \int_0^z -\frac{\log(1-t)}{t}\, dt = \sum_{n=1}^{\infty} \frac{z^n}{n^2}$$

Shadow coloring determines these and we set

Vol 

$$= D(z_N) + D(z_S) - D(z_W) - D(z_E)$$

z_i are explicit rational functions of g_i, L_i, a_j

- We can now assign $\mathrm{Vol}(D, g_i, u_j) \in \mathbb{R}$ to any shadow colored tangle diagram

Thm. This is a tangle invariant $+ \varphi$

- When D is a link diagram
  - It is gauge invariant (can shift u_j globally or conjugate g_i globally)
  - It recovers the volume of the induced hyperbolic metric

Even though $V(g, u)$ is 1-dim'l we have a nontrivial geometric invariant

More abstract:

For each D have decorated shadowed representation variety

$$\mathcal{Y}(D)$$

and section Vol of trivial bundle $\mathbb{R} \times \mathcal{Y}(D)$

For D = link diagram get gauge-invariant section,

So can pass to bundle over decorated character variety

$$\mathrm{Vol} \in \Gamma\left(\mathbb{R} \times \mathcal{X}(D)\right)$$

Why do we need line bundles?

M = closed 3-mfld

Def: $\rho : \pi_1(M) \to SL_2(\mathbb{C})$, A flat sl_2 connection with holonomy ρ

$$CS(\rho) = \frac{1}{4\pi i} \int_M \operatorname{tr} A \wedge dA + \frac{2}{3} \operatorname{tr} A \wedge A \wedge A \quad (\operatorname{mod} 2\pi i)$$

Complex Chern-Simons invariant. $\rho \rightsquigarrow$ hyperbolic structure $\rightsquigarrow$ metric g on M

$$\operatorname{Re} CS(\rho) = \frac{\text{volume}}{2\pi} \text{ of } g$$

$\operatorname{Im} CS(\rho) = SO(3)$ CS invariant of frame field of g

$$\mathcal{I}(\rho) = e^{CS(\rho)} \in \mathbb{C}^\times \quad \text{well-defined}$$

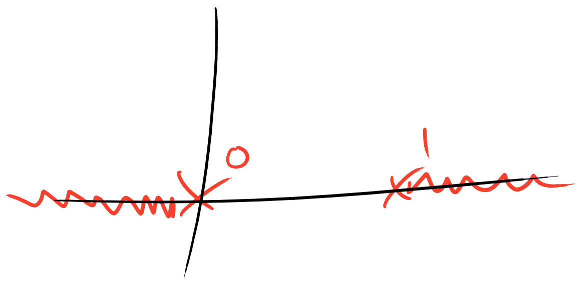
For a single tetrahedron τ w/ shape z,

$$CS(\zeta^0, \zeta^1) = Li_2(e^{2\pi i \zeta}) + \frac{2\pi i}{2}\zeta^0 \zeta^1 + \frac{1}{2}\zeta^0 \log(1 - e^{2\pi i \zeta^0})$$

lifted dilogarithm

$$e^{2\pi i \zeta^0} = z, \quad e^{2\pi i \zeta^1} = \frac{1}{1-z}$$

Really defined on cover of $\mathbb{C} \setminus \{0,1\}$



(ζ^0, ζ^1) flattening

Thm [Marché] These are boundary conditions for the flat connection.

Thm [Neumann] If you choose flattenings coherently get well-defined $CS(M, \rho)$, M closed.

If $M = S^3 \setminus K$ has torus boundary, $CS(S^3 \setminus K, \rho)$ depends on boundary conditions!

Thm [Kirk–Klassen] For M with $\partial M = S^1 \times S^1$, e^{CS} is section of a line bundle

for more torus components, just tensor

$$\begin{array}{c} \mathcal{E} \\ \downarrow \\ Y(\partial M) \end{array}$$

decorated rep var of M

Thm [CMS] Can assign shadow-colored tangle

$$\tilde{I}(D) \text{ section of line bundle } \begin{array}{c} \mathcal{E} \\ \downarrow \\ Y(D) \end{array}$$

obeying composition/gluing of tangles so that

$\hat{I}(L)$ recovers CS invariant.

Concretely:

$s =$ log-decoration $=$ boundary condition

$\tilde{I}(L, \rho, s) \in \mathbb{C}^\times$ $\qquad s \in H^1(\partial(S^3 - L); \mathbb{C})$

logarithm of $\rho|_{\text{boundary}}$

$\tilde{I}$ depends on s but in very simple way

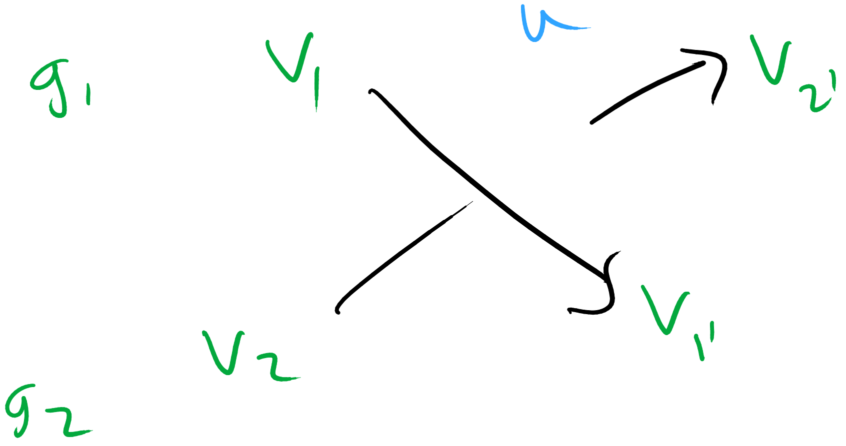
$$\tilde{I}(L, \rho, s') = e^{2\pi i \langle s' - s, s \rangle} \tilde{I}(L, \rho, s)$$

captured by line bundle.

What does this have to do with quantum invariants?

Thm [CMS, CMS-Reshetikhin]

One can assign shadow-colored diagrams **tensors**



$V_1 \otimes V_2 \longrightarrow V_{2'} \otimes V_{1'}$

obeying colored Reidemeister moves

$V_1 = V(g_1, u) \dots$

V_i rank N $\mathcal{U}_\zeta(\mathfrak{sl}_2)$-module, $\zeta = e^{\pi i/N}$

of Kac-de Concini type

Thm [CMS] These give an invariant

links are OK too

$$J_N(K, \rho, s) \in \mathbb{C}$$

K oriented framed knot in S^3

$\rho : \pi_1(S^3 \setminus K) \longrightarrow SL_2(\mathbb{C})$

decorated representation

s log-decoration

1. gauge-invariant

2. $J_N(K, \rho, s') = e^{-\frac{2\pi i \langle s'-s, \gamma\rangle}{N}} J_N(K, \rho, s)$

when $s'(\text{meridian}) \equiv s(\text{meridian}) \mod N\mathbb{Z}$

Abstractly: J_N is a section of a vector bundle over rep. var

3. when $\rho = (-1)^{N-1}$ is trivial $\quad J_N(K, (-1)^{N-1}, s_0) =$ Kashaev's invariant

canonical choice

4. [corollary of earlier CMS]

simple

$$J_2(K, \rho, s)\, J_2(\overline{K}, \overline{\rho}, \overline{s}) = \mathcal{K}(s)\, \tau(S^3 \setminus K, \rho)$$

mirror image

nonabelian Reidemeister torsion

This is a quantization in the sense of

invertible, classical theory $\rightsquigarrow$ non-invertible, quantum theory

Other reasons come from construction of J

dilogarithm $\rightsquigarrow$ ε-analogue of dilogarithm

Thm [Kac–de Concini] $U_\xi(sl_2)$ has big central subalgebra

$\xi^N = -1$

$Z_0 = \mathbb{C}[K^{\pm N}, E^N, F^N]$, Spec $Z_0 = SL_2(\mathbb{C})^*$

birational to $SL_2(\mathbb{C})$

an irrep V has character $\chi : Z_0 \to \mathbb{C}$

$\chi \Leftrightarrow \begin{bmatrix} \chi(K^N) & -\chi(E^N) \\ \chi(K^N F^N) & \cdots \end{bmatrix}$

encoding color $g \in SL_2(\mathbb{C})$.

Problem Unclear how to braid or even G-braid:

universal R-mat $\xi^{\frac{1}{2}H \otimes H} \sum_{n \geq 0} c_n E^n \otimes F^n$ diverges

since E, F can act non-nilpotently!

Thm [Kashaev–Reshetikhin] There is a projective braiding

$$V(\chi_1) \otimes V(\chi_2) \to V(\chi_{2'}) \otimes V(\chi_{1'})$$

w/ complicated transf. rule on χ_i. It satisfies RIII up to a scalar.

handled by shadows

Thm [CMS–Reshetikhin] This can be lifted to a braiding by keeping track of some logarithms

just like in classical CS invariant

Pf idea: Braiding is determined by outer auto $\mathcal{R} : \mathcal{U}_\xi^{\otimes 2} \to \mathcal{U}_\xi^{\otimes 2}$ (comes from conjugation by R-matrix). Map

$$R : V_1 \otimes V_2 \to V_1' \otimes V_2' \quad \text{is interfwiner}$$

$$R(u \cdot v) = \mathcal{R}(u) \cdot R(v) \qquad \begin{array}{l} v \in V_1 \otimes V_2 \\ u \in \mathcal{U}_\xi^{\otimes 2} \end{array}$$

To solve for matrix coeffs present $\mathcal{U}_\xi$ in terms of quantum torus $xy = \xi^2 yx$. Once solved can prove it satisfies R III exactly (not up to phase!) using extra choices of logarithms.

$\square$

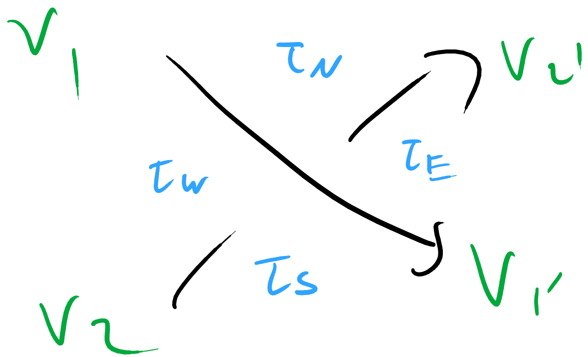
The data needed to determine the braiding are exactly analogous to the flattenings before

dilogarithm $\longleftrightarrow$ noncompact quantum dilogarithm

complex CS $\tilde{\mathcal{I}}$ $\qquad$ new inv. J_N

I was only able to solve the algebraic lifting property for these braid group representations by understanding (classical) complex CS theory

The braiding $c: V_1 \otimes V_2 \longrightarrow V_{2'} \otimes V_{1'}$ factors as



$$c: V_1 \otimes V_2 \longrightarrow V_{2'} \otimes V_{1'}$$

$$c = Z_E (Z_N \otimes Z_S) Z_W$$

with matrices Z_j defined using quantum dilogarithms and some parameters as in classical complex CS theory

Together these define J_N.

Bonus

Definition leads to a state-integral presentation of J_N much like those in

- one conjectured for complex CS by Yokota (et al?)
- Andersen–Kashaev's Teichmüller TQFT
- saddle-point approaches to Volume Conjecture