Towards quantized noncompact Chern-Simons theory

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Abstract

In this talk I will discuss recent joint work (with N. Reshetikhin) defining invariants $\mathcal{J}_N$ of knot (and link and tangle) exteriors with flat $\mathfrak{sl}_2$ connections. The construction is via a geometric version of the Reshetikhin-Turaev construction: it is algebraic and relies on the representation theory of quantum groups. Here I will instead focus on the properties of these invariants and explain why I think they are a good candidate for quantum Chern-Simons theory with noncompact gauge group $\mathrm{SL}_2(\mathbb{C})$. I will also discuss a connection with (and a generalization of) the Volume Conjecture.

1. What is quantum Chern-Simons theory?

Let G be a Lie group, real or complex, with Lie algebra $\mathfrak{g}$. (We will always take $G = \mathrm{SU}_2$ or $\mathrm{SL}_2(\mathbb{C})$.) Let M be a closed 3-manifold. We can identify a connection on a principal G-bundle over M with a $\mathfrak{g}$-valued 1-form A on M, and we define its Chern-Simons invariant by

$$\mathrm{CS}(A) = \frac{1}{C}\int_M \mathrm{tr}\left(A \wedge dA + \frac{2}{3}A \wedge A \wedge A\right)$$

for an appropriate constant C depending on G.

One can show that under gauge transformations $A \to A'$ we have (for the right value of C)

$$\mathrm{CS}(A') = \mathrm{CS}(A) + 2\pi k, k \in \mathbb{Z}$$

so

$$\mathcal{I}(A) = e^{i\,\mathrm{CS}(A)}$$

is well-defined under gauge transformations. In particular when A is flat $\mathcal{I}$ depends only on the holonomy, so it gives a function on the character variety of M (the space of representations $\rho : \pi_1(M) \to G$ modulo conjugation).

We want to understand the quantum field theory with action CS. This means considering the path integral/partition functions

$$\mathcal{Z}_k(M) = \int_{A \in \Omega^1(M,\mathfrak{g})} e^{ik\,\mathrm{CS}(A)} \mathcal{D}A$$

This is a formal object: it is difficult or impossible to assign a measure $\mathcal{D}$ to an infinite-dimensional manifold $\Omega^1(M, \mathfrak{g})$ that makes this integral exist. (Here k is an integer called the *level*.)

We can use the path integral as a heuristic to define a collection of related topological invariants called a *topological quantum field theory (TQFT)*:

- A closed manifold M gets a complex number $\mathcal{Z}_k(M)$.
- A closed surface Σ gets a vector space $\mathcal{Z}_k(\Sigma)$.
- A cobordism $Y : \Sigma_0 \to \Sigma_1$ gets a linear map $\mathcal{Z}_k(Y) : \mathcal{Z}_k(\Sigma_0) \to \mathcal{Z}_k(\Sigma_1)$ respecting composition
- Disjoint unions become tensor products

We can also allow (oriented, framed) knots and links inside M starting and ending on the various boundary components.

Famously Witten showed [Wit89] (at a physical level of rigor) how to interpret quantum Chern-Simons theory as a TQFT $\mathcal{Z}_k$ containing (the evaluation at certain roots of unity of)

[Wit89] E. Witten, "Quantum field theory and the Jones polynomial" the Jones polynomial and related invariants: the colored Jones polynomials are various levels of $\mathfrak{su}_2$, the HOMFLY-PT polynomial is $\mathfrak{su}_N$, and so on. The vector space $\mathcal{Z}_k(\Sigma)$ assigned to a surface Σ arises from geometric quantization of the moduli space $\mathfrak{X}(\Sigma)$ of flat connections on Σ; because we are using a compact group $\mathfrak{X}(\Sigma)$ has finite (symplectic) volume, so $\mathcal{Z}_k(\Sigma)$ is finite-dimensional.

Soon afterwards Reshetikhin and Turaev showed [RT91] how to rigorously construct $\mathcal{Z}_k$ using algebraic methods.[1] They explain how to construct a TQFT from an algebraic object called a *modular tensor category*.[2] One can construct these using a *quantum group* $\mathcal{U}_q(\mathfrak{g})$, a q-analogue of the universal enveloping algebra of $\mathfrak{g}$. For q a root of unity a modification of the category of $\mathcal{U}_q$-modules gives the required category. The order of the root of unity is related to the level k.

[RT91] N. Reshetikhin and V. G. Turaev, "Invariants of 3-manifolds via link polynomials and quantum groups". DOI

[1] Witten's approach can be made rigorous (at least in some cases) via geometric quantization of moduli spaces of flat connections on surfaces, but it is technically difficult and only appeared later.

[2] Part of the data of a TQFT are representations of the mapping class groups of all (oriented) surfaces, in particular of the modular group $\mathrm{SL}_2(\mathbb{Z}) = \mathrm{Mod}(T^2)$.

This is the approach we take. The Reshetikhin-Turaev construction starts from links in S^3 and uses surgery along these to extend to the general case. For $\mathrm{SL}_2(\mathbb{C})$ we have constructed invariants of link (exteriors) in S^3; I expect the extension to general manifolds via surgery will work but this is not done yet.

2. Complex Chern-Simons

We want to pass from the well-understood case $G = \mathrm{SU}_2$ to the noncompact, complex gauge group $\mathrm{SL}_2(\mathbb{C})$. Complex quantum Chern-Simons is not pinned down fully even by physics standards (as far as I know) and it is likely there are multiple reasonable answers for what it is. One significant issue is that because $\mathrm{SL}_2(\mathbb{C})$ is noncompact we expect the vector spaces assigned to surfaces to be infinite dimensional. However, it seems that the full theory is graded by a choice of background flat connection:

$$\mathcal{Z}(Y) = \bigoplus_{\rho \in \mathfrak{X}(Y)} \mathcal{Z}(Y, \rho)$$

with finite-dimensional pieces. In particular, we expect not a TQFT but a geometric quantum field theory[3] assigning invariants to Y plus a choice of (gauge class of) flat connection, i.e. of representations ρ into $\mathrm{SL}_2(\mathbb{C})$ modulo conjugation. We call the space of these the *character variety* $\mathfrak{X}(Y)$ of Y.

[3] Turaev calls these homotopy quantum field theories [Tur10].

To understand our quantum invariants it will be helpful to know a few more things about the ordinary $\mathrm{SL}_2(\mathbb{C})$ Chern-Simons invariant, which we normalize so

$$\mathrm{CS}(A) \in \mathbb{C}/(2\pi)^2 i\mathbb{Z} \text{ so } \mathcal{I}(A) = e^{\mathrm{CS}(A)/2\pi} \in \mathbb{C}^\times$$

When A is flat we can think of $\mathcal{I}(A)$ as a conjugation-invariant function of the holonomy $\rho = \mathrm{Hol}(A) : \pi_1(M) \to \mathrm{SL}_2(\mathbb{C})$. Since $\mathrm{Isom}(\mathbb{H}^3) = \mathrm{PSL}_2(\mathcal{C})$, ρ determines a (possibly degenerate) metric g of negative curvature on M, and it is a theorem of Yoshida [Yos85] that

$$\mathrm{CS}(\rho) = \mathrm{Vol}(g) + i\,\mathrm{CS}_{\mathfrak{su}_2}(g)$$

[Yos85] T. Yoshida, "The η-invariant of hyperbolic 3-manifolds". DOI

where the second term is the Chern-Simons invariant of the $\mathfrak{su}_2$ connection corresponding to the frame field of g. We know the hyperbolic volume is a strong invariant, so we hope the same is true for its quantization.

For manifolds with boundary there are boundary conditions to deal with, but for torus boundary components they are not too hard to handle [KK93]. Let $Y = S^3 \setminus \nu(K)$ be a knot exterior, for notational simplicity.[4] $\partial(Y) = T^2$ so $\pi_1(\partial Y) \cong \mathbb{Z}^2$ is abelian. Choosing a meridian $\mathfrak{m}$ and longitude $\mathfrak{l}$ gives a basis, and for representation $\rho : \pi_1(Y) \to \mathrm{SL}_2(\mathbb{C})$ the matrices $\rho(\mathfrak{m})$ and $\rho(\mathfrak{l})$ have a common eigenline (generically, two). A choice of eigenline is called a *decoration* of ρ. The decoration determines a basis with

$$\rho(\mathfrak{m}) = \begin{bmatrix} m & * \\ 0 & m^{-1} \end{bmatrix} \text{ and } \rho(\mathfrak{m}) = \begin{bmatrix} \ell & * \\ 0 & \ell^{-1} \end{bmatrix}$$

and in particular gives preferred eigenvalues m, ℓ of $\mathfrak{m}, \mathfrak{l}$.

[KK93] P. Kirk and E. Klassen, “Chern-Simons invariants of 3-manifolds decomposed along tori and the circle bundle over the representation space of T^2”. DOI

[4] This discussion works just as well for arbitrary 3-manifolds with boundary components all tori.

Any flat connection is gauge-equivalent to a constant one $\mu dx + \lambda dy$ near the boundary, where $e^\mu = m, e^\lambda = \ell$. We call μ, λ a *log-decoration* of ρ. The Chern-Simons invariant *does* depend on the log-decoration, but in a simple way:

$$\mathcal{I}(Y, \rho, \mu + 2\pi i a, \lambda + 2\pi i b) = e^{a\lambda - \mu b} \mathcal{I}(Y, \rho, \mu, \lambda)$$

Identifying the log-decorations with cohomology classes $\mathfrak{s}, \mathfrak{s}'$ we can write this as

$$\mathcal{I}(Y, \rho, \mathfrak{s}') = e^{\langle \mathfrak{s}' - \mathfrak{s}, \mathfrak{s} \rangle} \mathcal{I}(Y, \rho, \mathfrak{s})$$

for the pairing induced by the homology intersection pairing.

We can give this a more abstract definition. The space of decorated $\mathrm{SL}_2(\mathbb{C})$ representations of $\pi_1(T^2)$ modulo conjugation (the *decorated character variety*) $\mathfrak{X}^\delta(T^2)$ is $\mathbb{C}^\times \times \mathbb{C}^\times$, with coordinates (m, ℓ) as above. We can define a line bundle E over $\mathfrak{X}^\delta(T^2)$ whose fiber over (m, ℓ) is $\mathbb{C}$-valued functions of logarithms μ, λ transforming like $\mathcal{I}$ does above, and then we can think of $\mathcal{I}$ as taking values in the pullback of E to the decorated representation (in fact, character) variety of Y.

3. Quantum invariants

Our invariants $\mathcal{J}_N$ come from a modification of the Reshetikhin-Turaev construction. The original construction comes from interpreting a braid (or tangle, more generally) diagram as a linear map. A strand is assigned a vector space V, parallel strands to tensor products of the Vs, and a crossing to a linear map

$$c : V \otimes V \to V \otimes V$$

satisfying braid relations. One can obtain such a vector space as a module over the quantum group $\mathcal{U}_q(\mathfrak{g})$. For generic q the representation theory of $\mathcal{U}_q$ is just like for ordinary Lie algebras: we can classify everything in terms of highest weights. If V is (the analogue of) the defining rep of $\mathfrak{sl}_2$ we get the Jones polynomial; if it is a higher-dimensional one we get colored Jones polynomials, and so on. We do a G-graded version of this, with the gradings corresponding to holonomies.

Theorem 1 (CMS, Reshetikhin). For each integer $N \geq 2$ there is an invariant

$$\mathcal{J}_N(Y, \rho, \mu, \lambda)$$

[5] The definition still works for links, but for a technical reason we need to change normalization to make the invariant well-defined, and in this normalization the discussion about vector bundles doesn't quite work for boundary-parabolic representations.

of a knot exterior Y and a log-decorated representation $\rho : Y \to \mathrm{SL}_2(\mathbb{C})$.[5] It depends on the log-decoration as

$$\mathcal{J}_N(Y, \rho, \mu + 2\pi i N a, \lambda + 2\pi i b) = e^{\frac{1}{N}(\mu b - N a \lambda)} \mathcal{J}_N(Y, \rho, \mu, \lambda)$$

We think of $\mathcal{J}_N$ as a twisted or nonabelian version of the (colored) Jones polynomial. Specifically, when ρ is the trivial representation we recover the Nth colored Jones polynomial evaluated at $q = -\xi^{-1}$, $\xi = e^{\pi i/N}$, also known as the *Kashaev invariant* [Kas95; Kas97; MM01].[6] ⌟

[Kas95] R. M. Kashaev, "A link invariant from quantum dilogarithm". arXiv DOI

[Kas97] R. M. Kashaev, "The hyperbolic volume of knots from the quantum dilogarithm". arXiv DOI

[MM01] H. Murakami and J. Murakami, "The colored Jones polynomials and the simplicial volume of a knot". arXiv DOI

[6] Even in the geometrically trivial limit we are extending past the $\mathfrak{su}_2$ case: at the level corresponding to N the WRT invariants give the values at $1, \xi, \xi^2, \dots, \xi^{N-2}$, not $\xi^{N-1} = -\xi^{-1}$.

Note that there is now a dependence on the class of μ modulo $2\pi i N$. We can think of $\mathcal{J}_N$ as taking values in a rank N vector bundle: given ρ the values on any log-decoration are determined by the N-component vector

$$\mathcal{J}_N(\mu), \mathcal{J}_N(\mu + 2\pi i), \dots, \mathcal{J}_N(\mu + 2\pi i(N-1)).$$

This passage from the line bundle where $\mathcal{I}$ is valued to the related higher-rank vector bundle where $\mathcal{J}_N$ is valued is natural in the context of quantization.

We are still studying what this invariant really means. We do know it is nontrivial:

Theorem 2 (CMS). The product of the $N = 2$ invariant and the value on the mirror image knot

$$\mathcal{J}_2(Y, \rho, \mathfrak{s}) \mathcal{J}_2(\overline{Y}, \overline{\rho}, \overline{\mathfrak{s}}) = \tau(Y, \rho) \mathcal{K}(\mathfrak{s})$$

is the Reidemeister torsion of Y twisted by ρ times a simple normalization factor depending only on the log-decoration. ⌟

Proof. The hard part of this theorem is the main result of [McP22]. Determining the normalization factor $\mathcal{K}$ is straightforward after that. □

[McP22] C. McPhail-Snyder, "Holonomy invariants of links and nonabelian Reidemeister torsion". arXiv DOI

I will now describe how to define $\mathcal{J}_N$. Due to work of Kac and de Concini [DK90] it is known that for $q = \xi = e^{\pi i/N}$ a root of unity $\mathcal{U}_\xi(\mathfrak{sl}_2)$ has a large central subalgebra $\mathcal{Z}_0$. It is a commutative Hopf algebra, which means it is the algebra of functions on an algebraic group, which we call $\mathrm{SL}_2(\mathbb{C})^*$. This group is closely related to $\mathrm{SL}_2(\mathbb{C})$ (they are birational, for example). Furthermore by Schur's Lemma simple $\mathcal{U}_\xi$-modules are graded by $\mathrm{SL}_2(\mathbb{C})^*$. The grading encodes the $\mathrm{SL}_2(\mathbb{C})$ representation ρ of the knot complement [KR05].

[DK90] C. De Concini and V. G. Kac, "Representations of quantum groups at roots of 1"

[KR05] R. Kashaev and N. Reshetikhin, "Invariants of tangles with flat connections in their complements". arXiv DOI

We can now try to repeat the RT construction in a $\mathrm{SL}_2(\mathbb{C})^*$-graded way. The hard part is defining the braiding, which is now a map $V_1 \otimes V_2 \to V_{2'} \otimes V_{1'}$ between modules in different gradings, so we really need a family of braidings parametrized by the coordinates coming from $\mathcal{Z}_0$. These coordinates are geometrically natural: they turn out to be the (deformed) Ptolemy coordinates of the octahedral decomposition of the tangle complement. Usually the braiding would be given by the action of the universal R-matrix of $\mathcal{U}_q$, but it doesn't converge! We can still extract a *projective* (family of) braidings $\tilde{c}$ but for useful invariants we need to lift them to a genuine braiding.

To accomplish this we solve for the matrix coefficients explicitly by using a nonstandard presentation of $\mathcal{U}_\xi$ coming from quantum cluster algebras. Using this presentation we can explicitly solve for the braiding, which factors into 4 terms, one for each corner of the crossing. We can also interpret these as coming from the 4 flips defining the braiding (in terms of triangulations of surfaces) or the 4 ideal tetrahedra at the crossing in the octahedral decomposition (natural in hyperbolic geometry). Each term is given by a quantum dilogarithm, a q-analogue of the dilogarithm

$$\operatorname{Li}_2(z) = \int_1^z \frac{-\log(1-t)}{t} dt$$

used in the computation of the classical $\mathrm{SL}_2(\mathbb{C})$ Chern-Simons invariant $\mathcal{I}$. This function has branch cuts corresponding to both $\log z$ and $\log(1-z)$ and to compute $\mathcal{I}$ we need to make coherent choices of branches of these logarithms; Neumann explained how to do this [Neu04].

[Neu04] W. D. Neumann, “Extended Bloch group and the Cheeger-Chern-Simons class”. arXiv DOI

These logarithms *also* show up in the quantum case: in order to compute the matrix coefficients we need to choose a number of Nth roots. These are local, not global choices (they depend on the choice of a knot diagram) and we then need to to check our answer is independent of these or at least depends in a simple way. This problem is directly analogous to the algebraic computation of the classical Chern-Simons invariant (via the method of Neumann), and I was only able to solve it by understanding this analogy. Once this connection is made one can show that the braidings are either independent of these choices, or transform in a way corresponding to the log-decoration dependence.

Another way to say this: When computing the $\mathrm{SL}_2(\mathbb{C})$ Chern-Simons invariant in terms of a triangulation one needs to keep track of the boundary conditions for the flat connection. There is a framework for doing this in terms of *flattenings* [Neu04] via the *Ptolemy coordinates* [Zic09]. While there are not (yet) any explicit flat connections in the construction of $\mathcal{J}_N$ the flattenings and Ptolemy coordinates show up in an essential way.

[Zic09] C. K. Zickert, “The volume and Chern-Simons invariant of a representation”. arXiv DOI

4. State integrals and the volume conjecture

Another connection to complex Chern-Simons theory comes from the volume conjecture. Since Kashaev’s invariant comes from $\mathcal{J}_N$ evaluated at the trivial representation,[7] we can state the complexified volume conjecture as:

[7] In this case log-decorations are just logarithms of ± 1 and there’s a specific choice to make to get the Kashaev invariant

Conjecture 1 (Kashaev, Murakami). Let Y the exterior be a hyperbolic knot in S^3 and ρ_{hyp} (a lift of) its hyperbolic structure. Then there is a polynomial p (depending on Y) so that

$$\lim_{N\to\infty} \mathcal{J}_N(Y,1) \sim p(N^{1/2})\,\mathcal{I}(\rho_{\text{hyp}})^N$$

i.e.

$$\lim_{N\to\infty} \log \mathcal{J}_N(Y,1) \sim \frac{N}{2\pi}\left(\operatorname{Vol}(g) + i\,\mathrm{CS}_{\mathfrak{su}_2}(g)\right) + \text{lower-order terms}$$

One can be precise about what the lower-order terms are. ⌟

This conjecture is known for hyperbolic knots up to 7 crossings. Here we are writing the conjecture in terms of the value of $\mathcal{J}_N$ at a trivial flat connection, butit can be defined entirely algebraically [Kas97] and in terms of colored Jones polynomials [MM01]. As such it is surprising because it says that an invariant with an entirely algebraic definition can determine the

[Kas97] R. M. Kashaev, “The hyperbolic volume of knots from the quantum dilogarithm”. arXiv DOI

[MM01] H. Murakami and J. Murakami, “The colored Jones polynomials and the simplicial volume of a knot”. arXiv DOI hyperbolic volume. Because of this the conjecture would show that finite-type invariants detect the unknot.

It has been known for a while that the Volume Conjecture has something to do with complex Chern-Simons theory. Here's a heuristic version of the argument [Yok00].

[Yok00] Y. Yokota, "On the volume conjecture for hyperbolic knots". arXiv

An ideal hyperbolic tetrahedron is one with geodesic edges and its vertices on the boundary at infinity $\mathbb{C}P^1 = \partial\mathbb{H}$ of hyperbolic space. Up to congruence these are classified by a *shape parameter* $z \in \mathbb{C}P^1$, which is the cross-ratio of the vertices. When the tetrahedron is geometrically nondegenerate $z \in \mathbb{C} \setminus \{0, 1\}$. Given a triangulation of Y one can consider the space $\mathfrak{T}$ of (logarithms of) shape parameters of the triangulation. When these satisfy *gluing equations* on the edges the tetrahedra give a coherent hyperbolic structure on Y. One can write down an action[8] $\mathcal{S} : \mathfrak{T} \to \mathbb{C}$ on this space (which is roughly the Chern-Simons functional) whose critical points are solutions of the gluing equations (i.e. flat connections). The function is assembled from dilogarithms.

[8] In the literature this is usually called a "potential" but I think "action" is more accurate.

In many cases one can show that the Kashaev invariant is given by a contour integral over this space:

$$\mathcal{J}_N(Y, 1) \sim \int_\Gamma e^{NS(\vec{t})} d\vec{t}$$

Because ρ_{hyp} maximizes $\Re S$ among all critical points the Volume Conjecture comes down to

1. turning the Kashaev invariant into an integral over the parameter space $\mathfrak{T}$, and
2. showing that this integral is dominated by the saddle point contributions of S.

We can do 1 in fairly general circumstances for $\mathcal{J}_N$.

Theorem 3 (CMS). Let Y be the exterior of a knot K with diagram D. Then there is a function $S_N : \mathfrak{T} \to \mathbb{C}$ so that for any *non-pinched* representation $\rho : \pi_1(Y) \to \mathrm{SL}_2(\mathbb{C})$,

$$\mathcal{J}_N(Y, \rho, \mathfrak{s}) = \sum_{\vec{n} \in \mathbb{Z}^k} N^s \int_{[0,1]^k} e^{N\left[S_N(\vec{\beta}/N + \vec{t}) + 2\pi i \vec{t} \cdot \vec{k}\right]} d\vec{t}$$

for some s. Here the point $\vec{\beta}$ (but *not* S_N) depends on $\rho, \mathfrak{s}$. Furthermore $S_N \to S$ converges pointwise to the classical action. ⌟

Proof. It is immediate from the definition of $\mathcal{J}_N$ that one has

$$\mathcal{J}_N(Y, \rho, \mathfrak{s}) = N^{s'} \sum_{\vec{n} \in \mathbb{Z}^k} e^{N\left[S_N(\vec{\beta}/N + \vec{n}/N)\right]} d\vec{t}$$

for a function S_N assembled from quantum dilogarithms. This function is continuous (meromorphic, in fact!) so we can take its Fourier series. □

Conjecture 2. For any representation ρ one has

$$\lim_{N \to \infty} \mathcal{J}_N(Y, \rho) \sim p(N^{1/2})\, \mathcal{I}(\rho_{\text{hyp}})^N.$$ ⌟

Evidence. As $N \to \infty$ the dependence $\vec{\beta}/N$ on the initial representation ρ drops out, so the asymptotics only depend on S_N. This is converging to S (although not uniformly on its whole domain...) so the integral should be dominated by the saddle point asymptotics as before. □

The sum over integers is natural and even solves some problems. The gluing equations involve products of things being 1, so the logarithmic ones have a $2\pi i$ ambiguity having to do with branches of logarithms, and this sum is (roughly) over all possible branches. Being "non-pinched" is a technical non-degeneracy condition (the image of the two Wirtinger generators at each crossing should not share a fixed point). For reasonable diagrams (including all alternating diagrams, for example) the hyperbolic structure of a hyperbolic knot is not pinched.

The trivial representation is always pinched, so this theorem does not apply to the Kashaev invariant. However, it does achieve part (1) of the plan for proving the generalized volume conjecture. Unfortunately the lack of uniform convergence causes big problems for (2). Perhaps someone who is better at harmonic analysis than me can solve it.

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